Multiple Integrals

15.1 Double Integrals over Rectangles

B. Farman

Mathematics and Statistics
Louisiana Tech University

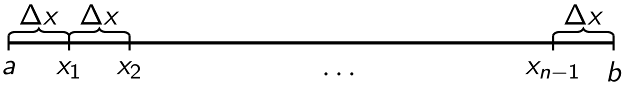
Calculus III

Review of the Definite Integral

Assume f is a continuous function on the interval $[a, b]$. Recall that a partition of $[a, b]$ is a choice of $n + 1$ points

$$x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

that determine n subintervals of width $\Delta x = (b - a)/n$



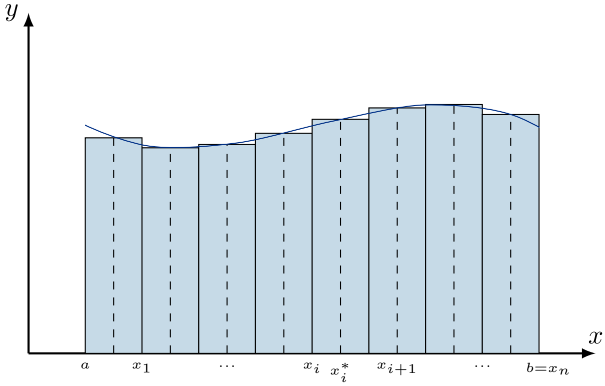
Review of the Definite Integral

Given a partition, we approximate the area under f on $[x_{i-1}, x_i]$ by choosing a sample point $x_{i-1} \le x_i^* \le x_i$ and constructing a rectangle of height $f\left(x_i^*\right)$. The sum of the areas of these rectangles, called the Riemann sum,

$$\sum_{i=1}^{n} f\left(x_i^*\right) \Delta x$$

approximates the area under f on $[a, b]$.

Review of the Definite Integral



Review of the Definite Integral

We define the area between f and the x-axis to be the definite integral

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

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Volumes and Double Integrals

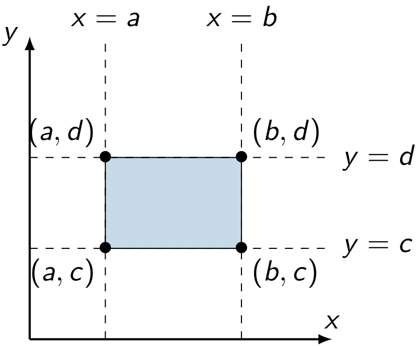
- Our goal is to mimic this construction to measure the volume of a solid bounded by a region in the xy-plane and the surface $z = f(x, y)$.
- General regions in the plane are difficult to handle, so we start with the two-dimensional analogue of closed sets, called rectangles.
- Rectangles are exactly what you think they are, but require a precise definition in order for us to work with them.

Volumes and Double Integrals

Definition (Rectangle)

A **rectangle** in $\mathbb{R}^2$ is the product of closed sets $[a, b]$ and $[c, d]$

$$R = [a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$$



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Volumes and Double Integrals

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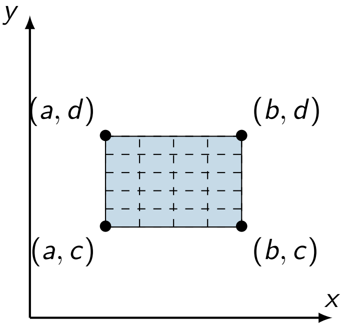
3. Take the product of these partitions

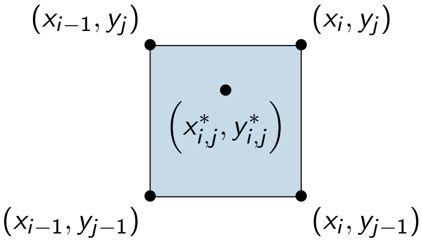
$$\{(x_i, y_j) \mid 0 \leq i \leq m,\, 1 \leq j \leq n\}$$

The product of these partitions creates a grid of sub-rectangles in the plane, each with area $\Delta A = \Delta x \Delta y$



Volumes and Double Integrals

For each sub-rectangle $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, choose a sample point $\left(x_{i,j}^*, y_{i,j}^*\right)$



Volumes and Double Integrals

- Construct a rectangular prism on the rectangle $R_{i,j}$ with height $f\left(x^*_{i,j}, y^*_{i,j}\right)$.

Volumes and Double Integrals

- Construct a rectangular prism on the rectangle $R_{i,j}$ with height $f\left(x_{i,j}^*, y_{i,j}^*\right)$.
- The volume,

$$V_{i,j} = f\left(x_{i,j}^*, y_{i,j}^*\right)\Delta x \Delta y = f\left(x_{i,j}^*, y_{i,j}^*\right)\Delta A,$$

approximates the volume of the solid with base the rectangle $R_{i,j}$ and top the portion of the surface

$$\{(x, y, f(x, y)) \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}$$

Definition (Double Riemann Sum)

We approximate the volume, V, of the solid bounded by the xy-plane and the surface $z = f(x, y)$ by the **double Riemann sum** of these volumes

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{i,j}^*, y_{i,j}^*\right) \Delta A = \sum_{j=1}^{n} \sum_{i=1}^{m} f\left(x_{i,j}^*, y_{i,j}^*\right) \Delta A$$

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Note the order of summation indicates summing volumes either by columns, then rows or by rows, then columns.

Volumes and Double Integrals

Definition (Double Integral)

The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right) \Delta A$$

provided the limit exists.

Volumes and Double Integrals

Volume

If $f(x, y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

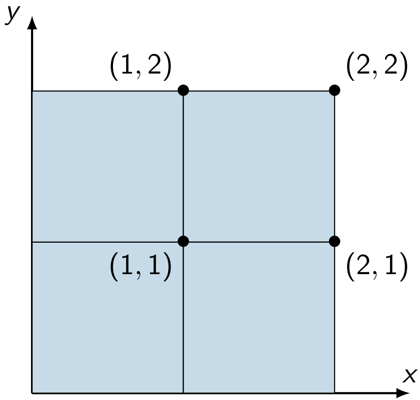
Exercise

Let $R = [0, 2] \times [0, 2]$ and $f(x, y) = 16 - x^2 - 2y^2$. Estimate $\iint_R f(x, y)\, dA$ using four equal squares with sample points the upper right corner of each square.

Volumes and Double Integrals

Solution (Part 1)

The partition of R with sample points marked is given below.



Solution (Part 2)

The height of the rectangular prism at each sample point is

$$
\begin{aligned}
f(1,1) &= 16 - (1)^2 - 2(1)^2 = 13 \\
f(1,2) &= 16 - (1)^2 - 2(2)^2 = 7 \\
f(2,1) &= 16 - (2)^2 - 2(1)^2 = 10 \\
f(2,2) &= 16 - (2)^2 - 2(2)^2 = 4
\end{aligned}
$$

Volumes and Double Integrals

Solution (Part 3)

As the area of each sub-rectangle is $\Delta A = 1$, the double Riemann sum is

$$\begin{aligned} V &\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{i,j}^{*}, y_{i,j}^{*}\right)\Delta A \\ &= f(1,1) + f(1,2) + f(2,1) + f(2,2) \\ &= 13 + 7 + 10 + 4 = 34 \end{aligned}$$

The Midpoint Rule

Theorem (Midpoint Rule)

If the double integral of $f(x, y)$ over the rectangle R exists, then

$$\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(\bar{x}_i, \bar{y}_j\right) \Delta A$$

where $\overline{x}_i$ and $\overline{y}_j$ are the midpoints of $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$,

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} \qquad \bar{y}_j = \frac{y_{j=1} + y_j}{2}$$

Iterated Integrals

Definition (Iterated Integral)

Assume f is a function of two variables and the double integral of f over $R = [a, b] \times [c, d]$ exists. An **iterated integral** has one of the following forms

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

Iterated Integrals

Evaluate the iterated integrals

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx \quad \text{and} \quad \int_1^2 \int_0^3 x^2 y \, dx \, dy.$$

Iterated Integrals

Solution (Part 1)

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 x^2 \left[\int_1^2 y^2 \, dy \right] dx = \int_0^3 \frac{4-1}{2} x^2 \, dx$$
$$= \frac{3}{2} \int_0^3 x^2 \, dx = \frac{3}{2} \left(\frac{1}{3} \right) \left(3^3 - 0^3 \right) = \frac{27}{2}$$

Iterated Integrals

Solution (Part 2)

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 y \left[\int_0^3 x^2 \, dx \right] dy = \int_1^2 \frac{27 - 0}{3} y \, dy$$

$$= 9 \int_1^2 y \, dy = 9 \left(\frac{1}{2} \right) \left(2^2 - 1^2 \right) = \frac{27}{2}$$

Iterated Integrals

Theorem (Fubini)

If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume f is bounded on R, f is discontinuous only on a finite number of smooth curves, an the iterated integrals exist.

Iterated Integrals

Exercise

Let $R = [0, 2] \times [1, 2]$. Evaluate

$$\iint_R (x - 3y^2)\, dA$$

Iterated Integrals

Solution

$$\iint_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx$$

$$= \int_0^2 \left[(2-1)x - 3\frac{8-1}{3} \right] dx = \int_0^2 (x - 7)\, dx$$

$$= \frac{4-0}{2} - 7(2-0) = -12$$

Iterated Integrals

Exercise

Let $R = [1, 2] \times [0, \pi]$. Evaluate $\iint_R y \sin(xy)\, dA$.

Iterated Integrals

Solution

$$\iint_R y\sin(xy) = \int_0^\pi y\left[\int_1^2 \sin(xy)\,dx\right]dy$$
$$= \int_0^\pi y\left(\frac{-\cos(2y)+\cos(y)}{y}\right)dy$$
$$= \int_0^\pi \cos(y)\,dy - \int_0^\pi \cos(2y)\,dy = 0$$

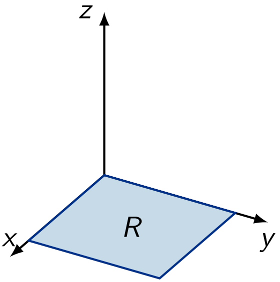
Exercise

Find the volume of the solid S bounded by $x^2 + 2y^2 + z = 16$, the planes $x = 2$ and $y = 2$, and the three coordinate planes.

Iterated Integrals

Solution (Part 1)

First, we sketch the region in the xy-plane. The intersection of $x = 2$ and $y = 2$ with the xy-plane form perpendicular lines that intersect the xz-plane at $(2, 0, 0)$ and the yz-plane at $(0, 2, 0)$, respectively. This tells us R is $[0, 2] \times [0, 2]$.



Solution (Part 2)

Rewrite the surface as $z = f(x, y) = 16 - x^2 - 2y^2$ so the desired volume is

$$\iint_R (16 - x^2 - 2y^2)\, dA$$

Iterated Integrals

Solution

$$\iint_R (16 - x^2 - 2y^2)\, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2)\, dx\, dy$$
$$= \int_0^2 \left(32 - \frac{8}{3} - 4y^2\right) dy = 64 - \frac{16}{3} - 4\left(\frac{8}{3}\right)$$
$$= 64 - \frac{48}{3} = 64 - 16 = 48$$

Iterated Integrals

Assume $R = [a, b] \times [c, d]$. If $f(x, y) = g(x)h(y)$, then

$$\begin{aligned}\iint_R f(x, y)\, dA &= \int_c^d \left[\int_a^b g(x)h(y)\, dx\right] dy \\ &= \int_c^d \left[h(y) \int_a^b g(x)\, dx\right] dy \\ &= \int_a^b g(x)\, dx \int_c^d h(y)\, dy\end{aligned}$$

Exercise

Compute the volume of the solid formed by $f(x, y) = \sin(x)\cos(y)$ over $R = [0, \pi/2] \times [0, \pi/2]$.

Iterated Integrals

Solution

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^{\frac{\pi}{2}} \sin(x)\,dx \int_0^{\frac{\pi}{2}} \cos(y)\,dy \\ &= \cos(0)\sin\left(\frac{\pi}{2}\right) \\ &= 1\end{aligned}$$

Average Value of a Function

Recall from §6.5

Definition

The **average value** of an integrable function, $f(x)$, on the interval $[a, b]$ is

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Average Value of a Function over a Plane Region

Definition

The **average value** of an integrable function f over a region R is

$$\overline{f} = \frac{1}{\text{area of } R} \iint_R f(x, y)\, dA.$$

Average Value

Exercise

Find the average value of the function $z = 2 - x - y$ over the square $R = [0, 2] \times [0, 2]$.

Average Value

Solution

$$\begin{aligned}
\overline{f} &= \frac{1}{4}\int_0^2 \left[\int_0^2 (2 - x - y)\,dx\right] dy \\
&= \frac{1}{4}\int_0^2 (4 - 2 - 2y)\,dy \\
&= \frac{8 - 4 - 4}{4} \\
&= 0
\end{aligned}$$