Multiple Integrals

15.7 Triple Integrals in Cylindrical Coordinates

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Calculus III

Cylindrical Coordinates

Cylindrical Coordinate System

Extending the polar coordinate system for $\mathbb{R}^2$ to include a height, (r, θ, z), yields a coordinate system for $\mathbb{R}^3$ known as the **cylindrical coordinate system**.

Cylindrical Coordinates

Converting from Cylindrical Coordinates

Given a point (r, θ, z) in the cylindrical coordinate system, we obtain the Cartesian coordinates using

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad z = z$$

Cylindrical Coordinates

Converting to Cylindrical Coordinates

Given a point (x, y, z) in Cartesian coordinates, we obtain the cylindrical coordinates using

$$r^2 = x^2 + y^2 \qquad \tan(\theta) = \frac{y}{x} \qquad z = z$$

provided $x \neq 0$.

Cylindrical Coordinates

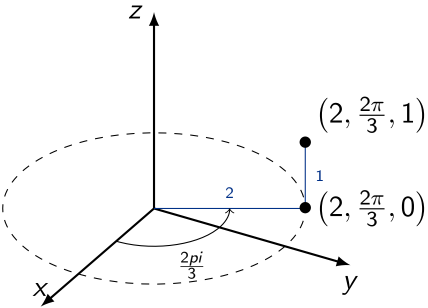
Exercise

1. Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its representation in Cartesian coordinates.
2. Find cylindrical coordinates of the point $(3, -3, -7)$.

Cylindrical Coordinates

Solution (Part 1)

In Cartesian coordinates, $(2, 2\pi/3, 1)$ is $(-1, \sqrt{3}, 1)$.



Cylindrical Coordinates

Solution (Part 2)

To convert $(3, -3, 7)$ to cylindrical, use

$$
\begin{aligned}
r &= \sqrt{18} = 3\sqrt{2} \\
\theta &= \arctan\left(\frac{-3}{3}\right) = \arctan(-1) = -\frac{\pi}{4} \\
z &= -7
\end{aligned}
$$

Make θ positive by adding 2π to obtain $\left(3\sqrt{2}, \frac{7\pi}{4}, -7\right)$.

Cylindrical Coordinates

Exercise

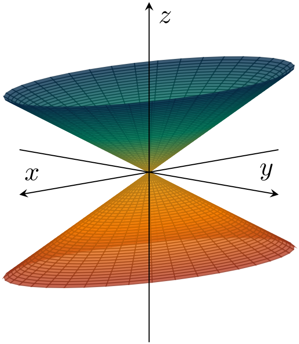
Describe the surface given in cylindrical coordinates by $z = r$.

Cylindrical Coordinates

Solution

The surface $z = r$ is the cone

$$z^2 = x^2 + y^2$$



Triple Integrals in Cylindrical Coordinates

Suppose E is a solid region of type 1 and D is its projection onto the xy-plane. If f is continuous on

$$E = \{(x, y, z) \mid (x, y) \in D,\ \rho(x, y) \leq z \leq \sigma(x, y)\}, \text{ and}$$
$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ g(\theta) \leq r \leq h(\theta)\},$$

then

$$\iiint_E f(x, y, z)\, dV$$
$$= \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} \int_{\rho(r\cos(\theta), r\sin(\theta))}^{\sigma(r\cos(\theta), r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z) r\, dz\, dr\, d\theta$$

Exercise

Let E be the solid under $z = 4 - x^2 - y^2$ and above $z = 0$. Evaluate $\iiint_E x^2 \, dV$.

Triple Integrals in Cylindrical Coordinates

Solution (Part 1)

The projection of E onto $z = 0$ is the disc of radius 2

$$D = \{(r, \theta) \mid 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$$

and thus

$$E = \left\{(r, \theta, z) \,\middle|\, 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 4 - r^2\right\}$$

Triple Integrals in Cylindrical Coordinates

Solution (Part 2)

$$\int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^3\cos^2(\theta)\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2\left(-r^5+4r^3\right)\cos^2(\theta)\,dr\,d\theta$$

$$=\frac{16}{3}\int_0^{2\pi}\cos^2(\theta)\,d\theta=\frac{16\pi}{3}$$

Triple Integrals in Cylindrical Coordinates

Exercise

A solid E lies within the part of the cylinder $x^2 + y^2 = 1$ with $y \geq 0$, below $z = 4$, and above $z = 1 - x^2 - y^2$. The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.

Triple Integrals in Cylindrical Coordinates

Solution (Part 1)

The projection of E onto the $z = 0$ is the half-disc

$$D = \{(r, \theta) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi\}$$

so that

$$E = \left\{(r, \theta, z) \;\middle|\; 0 \leq r \leq 1, 0 \leq \theta \leq \pi, 1 - r^2 \leq z \leq 4\right\}$$

Triple Integrals in Cylindrical Coordinates

Solution (Part 2)

The axis of the cylinder is the z-axis, so there exists a constant, k, such that the density is

$$\rho(r, \theta, z) = kr$$

Triple Integrals in Cylindrical Coordinates

Solution (Part 3)

$$m = \int_0^{\pi} \int_0^1 \int_{1-r^2}^4 kr^2 \, dz \, dr \, d\theta = k \int_0^{\pi} \int_0^1 \left(r^4 + 3r^2\right) dr \, d\theta$$
$$= \frac{6k}{5} \int_0^{\pi} d\theta = \frac{6k\pi}{5}$$

Triple Integrals in Cylindrical Coordinates

Exercise

Evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} \left(x^2 + y^2\right) \, dz \, dy \, dx$$

Triple Integrals in Cylindrical Coordinates

Solution

Recognize the projection of E onto $z = 0$ as the disc of radius 2, so in cylindrical coordinates:

$$\int_0^{2\pi} \int_0^2 \int_r^2 r^3 \, dz \, dr \, d\theta = \int_0^{2\pi} d\theta \int_0^2 \left(-r^4 + 2r^3\right) dr$$

$$= 2\pi \left(\frac{8}{5}\right) = \frac{16\pi}{5}$$