Multiple Integrals

15.4 Applications of Double Integrals

B. Farman

Mathematics and Statistics
Louisiana Tech University

Calculus III

Density and Mass

Definition (Mass)

Let ρ be the continuous density function of a thin plate or lamina defined over a region D in the xy-plane. The **mass** of the lamina is

$$m = \iint_D \rho(x, y)\, dA$$

Density and Mass

Definition (Electric Charge)

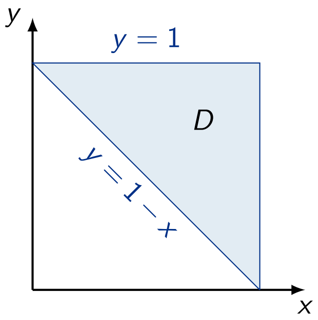
If an electric charge is distributed over a region D and the charge density at a point (x, y) is given by $\sigma(x, y)$, then the total **electric charge** Q is

$$Q = \iint_D \sigma(x, y)\, dA$$

Density and Mass

Exercise

The charge density on the region below is $\sigma(x, y) = xy$, measured in coulombs per square meter (C/m^2). Find the total charge.



Density and Mass

Solution

$$\int_0^1 \int_{1-x}^1 xy\, dy\, dx = \int_0^1 \left(x^2 - \frac{1}{2}x^3 \right) dx = \frac{5}{24}$$

Definition (Moment about the x-axis)

Let ρ be the continuous density function of a thin plate or lamina defined over a region D in the xy-plane. The **moment of the lamina about the x-axis** is

$$M_x = \iint_D y\rho(x, y)\, dA$$

Definition (Moment about the y-axis)

Let ρ be the continuous density function of a thin plate or lamina defined over a region D in the xy-plane. The **moment of the lamina about the y-axis** is

$$M_y = \iint_D x\rho(x, y)\, dA$$

Moments and Centers of Mass

Definition (Center of Mass)

Let ρ be the continuous density function of a thin plate or lamina defined over a region D in the xy-plane. The coordinates $(\bar{x}, \bar{y})$ for the **center of mass** of the lamina are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)\,dA \quad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)\,dA$$

where $m = \iint_D \rho(x,y)\,dA$.

Exercise

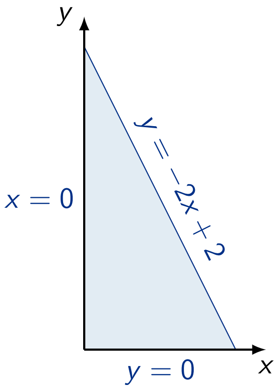
Find the center of mass of the triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$, and density function $\rho(x, y) = 1 + 3x + y$.

Moments and Centers of Mass

Solution (Part 1)

First, model the region

$$D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq -2x + 2\}$$



Solution (Part 2)

First, compute the mass.

$$m = \int_0^1 \int_0^{-2x+2} (1 + 3x + y)\, dy\, dx = \int_0^1 \left(-4x^2 + 4\right) dx = \frac{8}{3}$$

Moments and Centers of Mass

Solution (Part 3)

Compute the x-coordinate for the center of mass

$$\bar{x} = \frac{3}{8}\int_0^1 \int_0^{-2x+2} \left(x + 3x^2 + xy\right)\, dy\, dx$$

$$= \frac{3}{8}\int_0^1 \left(-4x^3 + 4x\right)\, dy = \frac{3}{8}$$

Moments and Centers of Mass

Solution (Part 4)

Compute the y-coordinate for the center of mass

$$\begin{aligned}\bar{y} &= \frac{3}{8}\int_0^1 \int_0^{-2x+2} (y + 3xy + y^2)\, dy\, dx \\ &= \frac{3}{8}\int_0^1 \left(\frac{10}{3}x^3 - 2x^2 - 6x + \frac{14}{3}\right) dx \\ &= \frac{11}{16}\end{aligned}$$

Moment of Inertia

Definition (Moment of Inertia about the x-axis)

Let ρ be the continuous density function of a thin plate or lamina defined over a region D in the xy-plane. The **moment of inertia of the lamina about the x-axis** is

$$I_x = \iint_D y^2 \rho(x, y)\, dA$$

Moment of Inertia

Definition (Moment of Inertia about the y-axis)

Let ρ be the continuous density function of a thin plate or lamina defined over a region D in the xy-plane. The **moment of inertia of the lamina about the y-axis** is

$$I_y = \iint_D x^2 \rho(x, y)\, dA$$

Moment of Inertia

Definition (Moment of Inertia about the Origin)

Let ρ be the continuous density function of a thin plate or lamina defined over a region D in the xy-plane. The **moment of inertia of the lamina about the origin** or the **polar moment of inertia** is

$$I_0 = I_x + I_y = \iint_D (x^2 + y^2)\rho(x, y)\, dA$$

Moment of Inertia

Exercise

Find the moments of inertia I_x, I_y, and I_0 of a homogeneous disc D with density $\rho(x, y) = \rho$, center the origin, and radius a.

Moment of Inertia

Solution

In polar, $D = \{(r,\theta) \mid 0 \leq r \leq a, 0 \leq \theta \leq 2\pi\}$

$$I_x = \rho \int_0^{2\pi} \cos^2(\theta)\, d\theta \int_0^a r^3\, dr = \frac{\rho a^4 \pi}{4}$$
$$I_y = \rho \int_0^{2\pi} \sin^2(\theta)\, d\theta \int_0^a r^3\, dr = \frac{\rho a^4 \pi}{4}$$
$$I_0 = I_x + I_y = \frac{\rho a^4 \pi}{2}$$

Definition (Radius of Gyration of a Lamina about an Axis)

The **radius of gyration of a lamina about an axis** is the number R such that $mR^2 = I$, where m is the mass of the lamina and I is the moment of inertia about the given axis. The radius of gyration of a lamina about the x-axis and y-axis are denoted by $\overline{\overline{y}}$ and $\overline{\overline{x}}$, respectively.

Moment of Inertia

Example

For the disc in the previous exercise, the radius of gyration, $\overline{\overline{y}}$, about the x-axis satisfies

$$\overline{\overline{y}} = \frac{I_x}{m} = \frac{\frac{a^2\pi\rho}{4}}{a^2\pi\rho} = \frac{a^2}{4}$$

so it follows that $\overline{\overline{y}} = a/2$.