Multiple Integrals

15.3 Double Integrals in Polar Coordinates

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Calculus III

Double Integrals in Polar Coordinates

- Suppose we want to find the volume of the solid bounded by the paraboloid

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and $z = 0$.

Double Integrals in Polar Coordinates

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$$z = 9 - x^2 - y^2$$
and $z = 0$.
- These surfaces intersect in the circle $x^2 + y^2 = 9$.

Double Integrals in Polar Coordinates

- The portion of the disc in the upper half-plane is

$$R = \left\{(x, y) \;\middle|\; -3 \leq x \leq 3,\, 0 \leq y \leq \sqrt{9 - x^2}\right\}$$

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- Take my word for it, you do not want to compute this integral.

Double Integrals in Polar Coordinates

Since we are working on a (semi)-circular region, we should switch to polar coordinates:

$$R = \{(x, y) \mid 0 \leq r \leq 3, \, 0 \leq \theta \leq \pi\}$$
$$z = 9 - r^2 \cos^2(\theta) - r^2 \sin^2(\theta) = 9 - r^2$$

and mimic the construction for rectangles in Cartesian Coordinates.

Double Integrals in Polar Coordinates

- Subdivide $[0, 3]$ into m sub-intervals via the choice of points

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of width $\Delta r = 3/m$.

Double Integrals in Polar Coordinates

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$$r_0 = 0 < r_1 < r_2 < \cdots < r_m = b$$

  of width $\Delta r = 3/m$.
- Subdivide $[0, \pi]$ into n subintervals via the choice of points

$$\theta_0 = 0 < \theta_1 < \theta_2 < \cdots < \theta_n = \pi$$

  of width $\Delta\theta = \pi/n$.

The product of these partitions provides a partition of R

$$\{(r_i, \theta_j) \mid 0 \le i \le m,\ 0 \le j \le n\}$$

Double Integrals in Polar Coordinates

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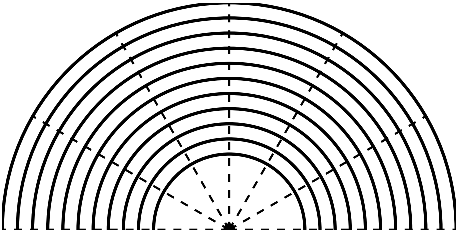
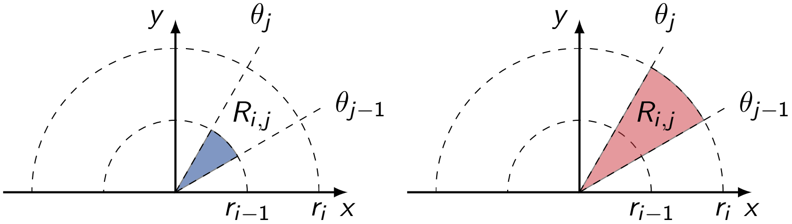
The partition subdivides R into mn polar rectangles

$$R_{i,j} = \{(r, \theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$



Double Integrals in Polar Coordinates

The area of the polar rectangle $R_{i,j}$ is given by subtracting the area of the sector of the circle with radius r_{j-1} with $\theta_{j-1} \leq \theta \leq \theta_j$ from the area of the sector of the circle with radius r_i with $\theta_{j-1} \leq \theta \leq \theta_j$



Double Integrals in Polar Coordinates

Recall the area of a circle of radius r is $r^2(2\pi - 0)/2$.

Double Integrals in Polar Coordinates

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$$\frac{r_{i-1}^2(\theta_j - \theta_{j-1})}{2} = \frac{r_{i-1}^2\Delta\theta}{2} \quad \text{and} \quad \frac{r_i^2(\theta_j - \theta_{j-1})}{2} = \frac{r_i^2\Delta\theta}{2}$$

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Hence the area of the polar rectangle $R_{i,j}$ is given by the formula

$$\begin{aligned}
\Delta A &= \frac{r_i^2 \Delta\theta - r_{i-1}^2 \Delta\theta}{2} \\
&= (r_i^2 - r_{i-1}^2)\Delta\theta \\
&= (r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta \\
&= \frac{r_i + r_{i-1}}{2}\Delta r \Delta\theta
\end{aligned}$$

Double Integrals in Polar Coordinates

- Simplify notation: $r_i^* = (r_i + r_{i-1})/2$, $\Delta A = r_i^* \Delta r \Delta \theta$.

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- Approximate the volume over $R_{i,j}$:

$$V_{i,j} \approx (9 - r_i^*)\,\Delta A = (9 - r_i^*) r_i^* \Delta r \Delta \theta$$

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- Sum to approximate the volume over R:

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} V_{i,j} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} (9 - r_i^*) r_i^* \Delta r \Delta\theta$$

Double Integrals in Polar Coordinates

Taking the limit as m and n tend to infinity gives the desired volume as an iterated integral,

$$\iint_R (9 - x^2 - y^2)\, dA = 2 \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (9 - r_i^*) r_i^* \Delta r \Delta\theta$$

$$= 2 \int_0^{\pi} \int_0^3 (9 - r^2) r\, dr\, d\theta = 2 \int_0^{\pi} d\theta \left[9 \int_0^3 r\, dr - \int_0^3 r^3\, dr \right]$$

$$= 2\pi \left[\frac{81}{2} - \frac{81}{4} \right] = \frac{81\pi}{2}$$

Double Integrals in Polar Coordinates

Theorem

If f is continuous on the polar rectangle

$$R = \{(r, \theta) \mid 0 \le a \le r \le b,\, \alpha \le \theta \le \beta\},$$

where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f\left(r\cos(\theta), r\sin(\theta)\right) r\, dr\, d\theta.$$

Double Integrals in Polar Coordinates

Exercise

Let R be the region in the upper half-plane bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Evaluate $\iint_R \left(3x + 4y^2\right) dA$.

Double Integrals in Polar Coordinates

Solution (Part 1)

First, convert to polar coordinates

$$
\begin{aligned}
R &= \{(r,\theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\} \\
f(r\cos(\theta), r\sin(\theta))r &= (3r\cos(\theta) + 4r^2\sin^2(\theta))r \\
&= 3r^2\cos(\theta) + 4r^3\sin^2(\theta)
\end{aligned}
$$

Double Integrals in Polar Coordinates

Solution (Part 2)

Now compute

$$\begin{aligned}\iint_R (3x+4y^2)\,dA &= \int_0^{\pi}\int_1^2 \left(3r^2\cos(\theta)+4r^3\sin^2(\theta)\right)dr\,d\theta\\ &= 7\int_0^{\pi}\cos(\theta)\,d\theta + \frac{15}{2}\int_0^{\pi}(1-\cos(2\theta))\,d\theta\\ &= \frac{15}{2}\int_0^{\pi}d\theta = \frac{15\pi}{2}\end{aligned}$$

Double Integrals in Polar Coordinates

Exercise

Evaluate

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)\,dy\,dx$$

by changing to polar coordinates.

Double Integrals in Polar Coordinates

Solution (Part 1)

First, recognize this integral as $\iint_R (x^2 + y^2)\, dA$ over the half-disc

$$R = \{(r, \theta) \mid 0 \leq r \leq 1,\, 0 \leq \theta \leq \pi\}$$

then compute

$$f(r\cos(\theta), r\sin(\theta)r = \left(r^2\cos^2(\theta) + r^2\sin^2(\theta)\right) r = r^3$$

Double Integrals in Polar Coordinates

Solution (Part 2)

Now integrate:

$$\iint_R (x^2 + y^2)\, dA = \int_0^{\pi} d\theta \int_0^1 r^3\, dr = \frac{\pi}{4}$$

Exercise

Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Double Integrals in Polar Coordinates

Solution

Write $f(x, y) = 1 - x^2 - y^2$. The paraboloid intersects the xy-plane in the circle of radius 1

$$R = \{(1, \theta) \mid 0 \leq \theta \leq 2\pi\}$$

and

$$f(r\cos(\theta), r\sin(\theta))r = \left(1 - r^2\cos^2(\theta) - r^2\sin^2(\theta)\right)r = (1 - r^2)r$$

Double Integrals in Polar Coordinates

Solution

Use the substitution $u = r^2$, $du/2 = r\,dr$ to compute

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{2\pi} d\theta \int_0^1 (1-r^2)r\,dr \\
&= 2\pi\left(\frac{1}{2}\right)\int_0^1 (1-u)\,du \\
&= \pi\left(\frac{1}{2}\right) = \frac{\pi}{2}
\end{aligned}$$

Double Integrals in Polar Coordinates

Theorem (General Polar Regions)

If f is continuous on the polar region

$$R = \{(r, \theta) \mid g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\},$$

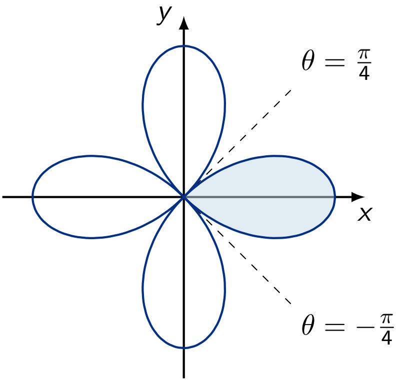
where $0 < \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos(\theta), r\sin(\theta))r\, dr\, d\theta.$$

Double Integrals in Polar Coordinates

Exercise

Find the area of the region, R, enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.



Solution

Write $R = \{(r, \theta) \mid 0 \le r \le \cos(2\theta), -\pi/4 \le \theta \le \pi/4\}$ and compute

$$A(R) = \iint_R dA = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_0^{\cos(2\theta)} r\,dr\,d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta)\,d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos(4\theta))\,d\theta = \frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta = \frac{\pi}{8}$$

Exercise

Find the volume of the solid under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Double Integrals in Polar Coordinates

Solution

The cylinder

$$x^2 + y^2 = 2x \iff (x-1)^2 + y^2 = 1$$

intersects $z = 0$ in the circle tangent to the origin through $(2, 0)$, which is the polar curve $r = 2\cos(\theta)$ that bounds the region

$$R = \{(r, \theta) \mid 0 \le r \le 2\sin(\theta), -\pi/2 \le \theta \le \pi/2\}$$

Double Integrals in Polar Coordinates

Solution

Recognize the height of the solid is given in polar coordinates by $z = r^2$, so the volume is

$$V = \iint_R (x^2 + y^2)\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} r^3\, dr\, d\theta = 4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4(\theta)\, d\theta$$

$$= 4 \left[\frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{3\pi}{2}$$