Multiple Integrals

15.2 Double Integrals over General Regions

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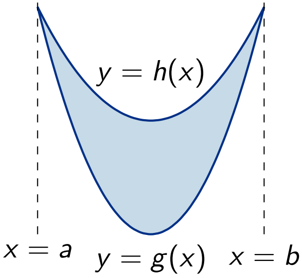
Calculus III

General Regions

Definition (Plane Region of Type I)

A region D in the xy-plane is said to be of **Type I** if there exist continuous functions g and h such that

$$D = \{(x, y) \mid a \leq x \leq b,\, g(x) \leq y \leq h(x)\}$$



General Regions

Theorem

Assume

$$D = \{(x, y) \mid a \leq x \leq b,\, g(x) \leq y \leq h(x)\}$$

is of type I. If f is continuous on D, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx$$

General Regions

Exercise

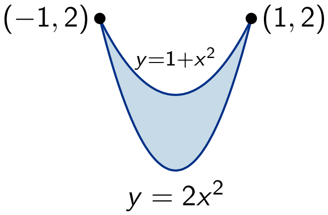
Let D be the region bounded by $y = 2x^2$ and $y = 1 + x^2$.
Evaluate $\iint_D (x + 2y)\, dA$.

General Regions

Solution (Part 1)

First, recognize D is a region of type I

$$D = \left\{ (x, y) \;\middle|\; -1 \leq x \leq 1,\ 1 + x^2 \leq y \leq 2x^2 \right\}$$



General Regions

Solution (Part 2)

$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} (x+2y)\,dy\,dx$$

$$= \int_{-1}^{1} \left[-3x^4 - x^3 + 2x^2 + x + 1\right] dx$$

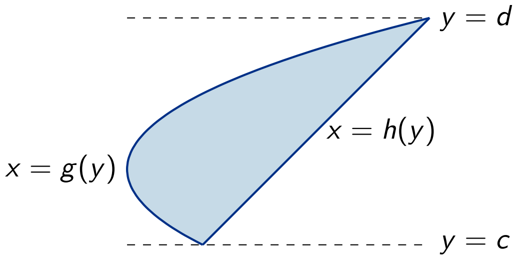
$$= -3\int_0^1 x^4\,dx + 2\int_0^1 x^2\,dx + \int_0^1 dx = \frac{32}{15}$$

General Regions

Definition (Plane Region of Type II)

A region D in the xy-plane is said to be of **Type II** if there exist continuous functions g and h such that

$$D = \{(x, y) \mid c \leq y \leq d,\ g(y) \leq x \leq h(y)\}$$



General Regions

Theorem

Assume

$$D = \{(x, y) \mid c \leq y \leq d,\, g(y) \leq x \leq h(y)\}$$

is of type II. If f is continuous on D, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy$$

General Regions

Exercise

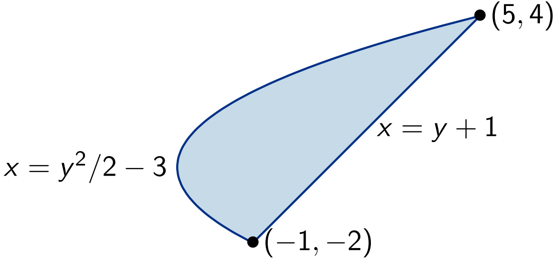
Let D be the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. Evaluate $\iint_D xy\, dA$.

General Regions

Solution (Part 1)

First, recognize D is a region of type II

$$D = \left\{ (x, y) \,\middle|\, -2 \leq y \leq 4,\ \frac{1}{2}y^2 - 3 \leq x \leq y + 1 \right\}$$



Solution (Part 2)

Now evaluate the iterated integral

$$\iint_D xy\, dA = \int_{-2}^{4} \int_{\frac{y^2}{2}-3}^{y+1} xy\, dx\, dy = 36$$

General Regions

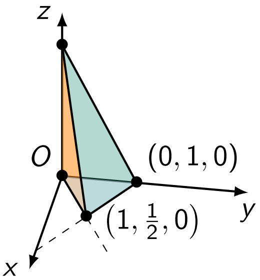
Exercise

Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

General Regions

Solution (Part 1)

First, observe the intersection of the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ with the plane $z = 0$ produces a triangle in the xy-plane



General Regions

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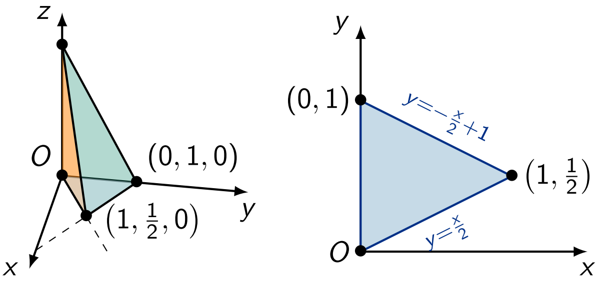

Figure: The tetrahedron (left) and its base (right).

General Regions

Solution

- The base of the tetrahedron is a region of type I

$$D = \left\{ (x, y) \,\middle|\, 0 \leq x \leq 1, \frac{x}{2} \leq y \leq -\frac{x}{2} + 1 \right\}$$

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- At each point (x, y) in D, the height of the solid is $f(x, y) = 2 - x - 2y$.

General Regions

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- The base of the tetrahedron is a region of type I

$$D = \left\{ (x, y) \,\middle|\, 0 \le x \le 1, \, \frac{x}{2} \le y \le -\frac{x}{2} + 1 \right\}$$

- At each point (x, y) in D, the height of the solid is $f(x, y) = 2 - x - 2y$.
- Therefore volume of the solid is $V = \iint_D f(x, y)\, dA$.

Solution (Part 3)

Now evaluate the iterated integral

$$V = \int_0^1 \int_{\frac{x}{2}}^{-\frac{1}{2}x+1} (2 - x - 2y)\, dy\, dx = \frac{1}{3}$$

Changing the Order of Integration

Exercise

Attempt to evaluate

$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$$

What goes wrong? How can we fix it?

Changing the Order of Integration

Solution (Part 1)

- We cannot find an elementary anti-derivative for $\sin(y^2)$.

Changing the Order of Integration

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- We *can* for $y\sin(y^2)$; use $u = y^2$, $du/2 = dy$:

$$\int y\sin(y^2)\,dy = \frac{1}{2}\int \sin(u)\,du = -\frac{1}{2}\cos(y^2) + C$$

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- This tells us we should attempt to reverse the order of integration.

Changing the Order of Integration

Solution (Part 2)

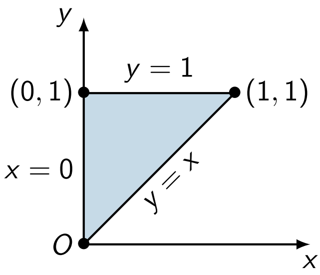
Draw the region, D, of type I over which we want to integrate.

Changing the Order of Integration

Solution (Part 2)

Draw the region, D, of type I over which we want to integrate. Rewrite as a region of type II:

$$D = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}$$



Changing the Order of Integration

Solution (Part 3)

Rewrite the integral

$$\iint_D \sin(y^2)\,dA = \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$
$$= \int_0^1 y\sin(y^2)\,dy$$
$$= \frac{1-\cos(1)}{2}$$

Properties of Double Integrals

Assume f and g are integrable on D, and c, m and M are constants.

Linearity

$$\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

$$\iint_D cf(x,y)\, dA = c \iint_d f(x,y)\, dA$$

Properties of Double Integrals

Area

The area of the region D is

$$A(D) = \iint_D 1 \, dA$$

Properties of Double Integrals

Order

If $g(x,y) \leq f(x,y)$ on D, then

$$\iint_D g(x,y)\,dA \leq \iint_D f(x,y)\,dA$$

In particular, if $m \leq f(x,y) \leq M$, then

$$mA(D) \leq \iint_D f(x,y)\,dA \leq MA(D)$$

Properties of Double Integrals

Additivity

If $D = D_1 \cup D_2$ and the regions D_1 and D_2 overlap only on the boundary, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

Properties of Double Integrals

Exercise

Let D be the disc of radius 2 about the origin. Estimate

$$\iint_D e^{\sin(x)\cos(y)}\,dA$$

Properties of Double Integrals

Solution

- Take the conservative bound $-1 \leq \sin(x), \cos(y) \leq 1$.

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- Hence $\frac{1}{e} \leq e^{\sin(x)\cos(y)} \leq e$.

Properties of Double Integrals

Solution

- Take the conservative bound $-1 \leq \sin(x), \cos(y) \leq 1$.
- Observe this implies $-1 \leq \sin(x)\cos(y) \leq 1$.
- Hence $\frac{1}{e} \leq e^{\sin(x)\cos(y)} \leq e$.
- Therefore

$$\frac{A(D)}{e} = \frac{4\pi}{e} \leq \iint_D e^{\sin(x)\cos(y)} \leq 4\pi e = A(D)e.$$