Partial Derivatives

14.2 Limits and Continuity

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Limits of Functions of Two Variables

Definition (Intuitive Definition of the Limit of a Function of Two Variables)

Suppose the multivariable function f is defined for all (x, y) near (a, b) except possibly at (a, b). If $f(x, y)$ can be made arbitrarily close to L for all (x, y) sufficiently close (but not equal) to (a, b), we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

and say the **limit of** f **as** (x, y) **approaches** (a, b) **equals** L.

Definition (Precise Definition of the Limit of a Function of Two Variables)

Let f be a function of two variables whose domain D include points arbitrarily close to (a, b). We say the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L if for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 \leq \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \epsilon$.

Showing That a Limit Does Not Exist

Theorem

If $f(x,y) \to L_1$ *as* $(x,y) \to (a,b)$ *along a path* C_1 *and* $f(x,y) \to L_2$ *as* $(x,y) \to (a,b)$, *where* $L_1 \neq L_2$, *then* $\lim_{(x,y)\to(a,b)} f(x,y)$ *does not exist.*

Showing That a Limit Does Not Exist

Exercise

Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

Showing That a Limit Does Not Exist

Solution (Part 1)

Consider the following two paths to $(0, 0)$

- Along the x-axis.

Showing That a Limit Does Not Exist

Solution (Part 1)

Consider the following two paths to $(0, 0)$

- Along the x-axis.
- Along the y-axis.

Showing That a Limit Does Not Exist

Solution (Part 2)

Every point along the x-axis has the form $(x, 0)$, so the limit reduces to

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$$

Showing That a Limit Does Not Exist

Solution (Part 3)

Every point along the y-axis has the form $(0, y)$, so the limit reduces to

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y\to 0} -\frac{y^2}{y^2}$$
$$= \lim_{y\to 0} -1$$
$$= -1$$

Showing That a Limit Does Not Exist

Solution (Part 4)

Since the limit of $f(x, y)$ as (x, y) approaches $(0, 0)$ along the x-axis is 1 and along the y axis is -1, we conclude that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

does not exist.

Showing That a Limit Does Not Exist

Exercise

Determine whether

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

exists.

Showing That a Limit Does Not Exist

Solution (Part 1)

This limit does not exist. To see this, consider the two paths to $(0, 0)$

- Along the line $y = x$.

Showing That a Limit Does Not Exist

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This limit does not exist. To see this, consider the two paths to $(0, 0)$

- Along the line $y = x$.
- Along the line $y = -x$.

Showing That a Limit Does Not Exist

Solution (Part 2)

Along the line $y = x$, the limit reduces to

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

Showing That a Limit Does Not Exist

Solution (Part 3)

Along the line $y = -x$, the limit reduces to

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} -\frac{x^2}{2x^2}$$
$$= -\frac{1}{2} \neq \frac{1}{2}$$

Showing That a Limit Does Not Exist

Exercise

Determine whether

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

exists.

Showing That a Limit Does Not Exist

Solution (Part 1)

This limit does not exist. To see this, consider the following two paths to $(0,0)$:

- C_1 traces the portion of the parabola $x = y^2$ above the x-axis.

Showing That a Limit Does Not Exist

Solution (Part 1)

This limit does not exist. To see this, consider the following two paths to $(0, 0)$:

- C_1 traces the portion of the parabola $x = y^2$ above the x-axis.
- C_2 traces the portion of the parabola $x = -y^2$ above the x-axis.

Showing That a Limit Does Not Exist

Solution (Part 2)

Along C_1, the limit reduces to

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

Showing That a Limit Does Not Exist

Solution (Part 3)

Along C_2, the limit reduces to

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} &= \lim_{y\to 0} -\frac{y^4}{2y^4} \\ &= \lim_{y\to 0} -\frac{1}{2} \\ &= -\frac{1}{2} \neq \frac{1}{2} \end{aligned}$$

Properties of Limits

Theorem

Assume $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{(x,y)\to(a,b)} g(x) = M$

1. $\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] + L + M.$
2. $\lim_{(x,y)\to(a,b)} [f(x,y) - g(x,y)] - L - M.$
3. $\lim_{(x,y)\to(a,b)} cf(x,y) = cL,\ c\ a\ constant.$
4. $\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = LM.$
5. $\lim_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \frac{L}{M},\ provided\ M \neq 0.$

Properties of Limits

Definition (Polynomial Function of Two Variables)

A **polynomial function** of two variables is a sum of terms of the form $cx^m y^n$, where c is a constant, and n and m are integers.

Definition (Rational Function of Two Variables)

A **rational function** of two variables is a ratio of two polynomials of two variables.

Properties of Limits

Theorem

If $p(x, y)$ is a polynomial, then

$$\lim_{(x,y)\to(a,b)} p(x, y) = p(a, b).$$

Properties of Limits

Theorem

If $p(x,y)$ *and* $q(x,y)$ *are polynomials and* $q(a,b) \neq 0$, *then*

$$\lim_{(x,y)\to(a,b)} \frac{p(x,y)}{q(x,y)} = \frac{p(a,b)}{q(a,b)}$$

Properties of Limits

Exercise

Evaluate the following limits

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$$

$$\lim_{(x,y)\to(-2,3)} \frac{x^2y + 1}{x^3y^2 - 2x}$$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2}$$

Properties of Limits

Solution (Part 1)

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 8 - 4 + 3 + 4 = 11$$

$$\lim_{(x,y)\to(-2,3)} \frac{x^2y+1}{x^3y^2-2x} = \frac{12+1}{-72+4} = -\frac{13}{68}$$

Properties of Limits

Solution (Part 2)

Observe that if the limit of $f(x, y) = (3x^2y)/(x^2 + y^2)$ as (x, y) approaches the origin, then its value **must** be the value of the limit along the x-axis. Since $f(x, 0) = 3x^2(0)/(x^2 + 0^2) = 0$ along the x-axis, this tells us the limit must be zero if it exists.

Solution (Part 3)

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If we believe the limit is zero, then we must show for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then

$$|f(x, y) - 0| = |f(x, y)| = \left| \frac{3x^2 y}{x^2 + y^2} \right|$$

Properties of Limits

Solution (Part 3)

If we believe the limit is zero, then we must show for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\begin{aligned} |f(x,y) - 0| &= |f(x,y)| = \left| \frac{3x^2 y}{x^2 + y^2} \right| \\ &= \frac{3x^2 |y|}{x^2 + y^2} \end{aligned}$$

Properties of Limits

Solution (Part 3)

If we believe the limit is zero, then we must show for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\begin{aligned} |f(x,y) - 0| &= |f(x,y)| = \left| \frac{3x^2 y}{x^2 + y^2} \right| \\ &= \frac{3x^2 |y|}{x^2 + y^2} \leq \frac{3x^2 |y|}{x^2} \end{aligned}$$

Properties of Limits

Solution (Part 3)

If we believe the limit is zero, then we must show for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\begin{aligned} |f(x,y) - 0| &= |f(x,y)| = \left| \frac{3x^2 y}{x^2 + y^2} \right| \\ &= \frac{3x^2 |y|}{x^2 + y^2} \leq \frac{3x^2 |y|}{x^2} \\ &= 3|y| < \epsilon \end{aligned}$$

Properties of Limits

Solution (Part 3)

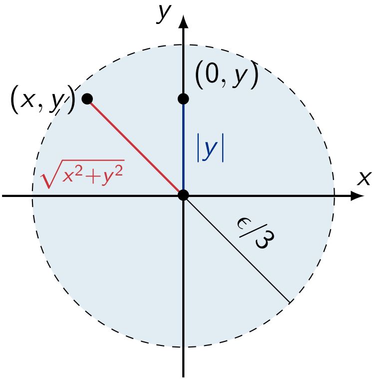
If we believe the limit is zero, then we must show for every $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < \sqrt{x^2 + y^2} < \delta$, then

$$\begin{aligned} |f(x,y) - 0| &= |f(x,y)| = \left| \frac{3x^2 y}{x^2 + y^2} \right| \\ &= \frac{3x^2 |y|}{x^2 + y^2} \leq \frac{3x^2 |y|}{x^2} \\ &= 3|y| < \epsilon \iff |y| < \frac{\epsilon}{3} \end{aligned}$$

Properties of Limits

Solution (Part 4)

Hence if we restrict our points, (x, y), to the interior of the disc of radius $\delta = \epsilon/3$, then the y-coordinate is guaranteed to be less than $\epsilon/3$ units from the origin:



Properties of Limits

Solution (Part 5)

Hence we have shown that if $0 < \sqrt{x^2 + y^2} < \epsilon/3$, then $|f(x, y)| \leq 3|y| < \epsilon$. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2} = 0$$

Continuity

Definition (Continuous Function of Two Variables)

A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say that f is **continuous on** D if f is continuous at every point (a, b) in D.

Continuity

Exercise

Where is the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

continuous?

Continuity

Solution

The function f is continuous on its domain

$$D = \{(x, y) \mid (x, y) \neq 0\}$$

Continuity

Exercise

Show the function

$$f(x,y) = \begin{cases} \frac{3x^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous.

Continuity

Solution

For all $(a, b) \neq (0, 0)$

$$\lim_{(x,y)\to(a,b)} f(x, y) = \lim_{(x,y)\to(a,b)} \frac{3x^2y}{x^2 + y^2} = \frac{3a^2b}{a^2 + b^2} = f(a, b)$$

because $(3x^2y)/(x^2 + y^2)$ is a rational function, and

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2 + y^2} = 0 = f(0, 0).$$

Continuity

Exercise

Where is the function $f(x, y) = e^{-(x^2+y^2)}$ continuous?

Solution

This function is continuous on $\mathbb{R}^2$ because $g(t) = e^{-t}$ is continuous, $p(x, y) = x^2 + y^2$ is continuous, and

$$f(x, y) = e^{-(x^2+y^2)} = g \circ p(x, y)$$

Continuity

Exercise

Where is the function $f(x, y) = \arctan\left(\frac{y}{x}\right)$ continuous?

Continuity

Solution

The rational function $r(x, y) = y/x$ is continuous on

$$D = \{(x, y) \mid x \neq 0\}$$

and the function $g(x) = \arctan(x)$ is continuous on $\mathbb{R}$, so

$$f(x, y) = g \circ r(x, y)$$

is continuous on D.

Functions of Three or More Variables

Definition (Limit)

Assume f is a defined on a subset D of $\mathbb{R}^n$. Write $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. We say $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{x} \in D$ and

$$0 < |\mathbf{x} - \mathbf{a}| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2} < \delta,$$

then $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$.

Functions of Three or More Variables

Definition (Continuous)

Assume f is a defined on a subset D of $\mathbb{R}^n$. The function f is **continuous at a** if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a}).$$