Partial Derivatives

14.1 Functions of Several Variables

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Calculus III

Functions of Two Variables

Definition (Function of Two Variables)

A **function of two variables** is a rule, f, that assigns to each ordered pair (x, y) in a set D a unique real number $f(x, y)$. The set D is the **domain** of f. The **range** of f is the set of real numbers that f takes on; that is, $\{f(x, y) \mid (x, y) \in D\}$.

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- The variables x and y are the **independent variables**.
- The variable z is the **dependent variable**.

Functions of Two Variables

Exercise

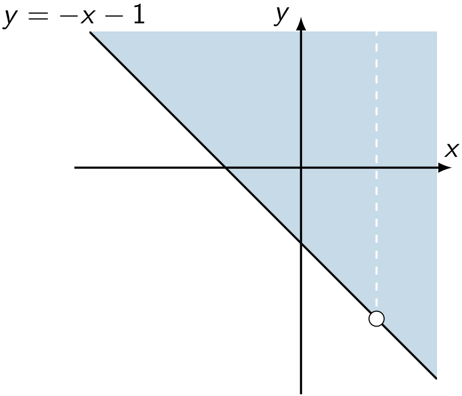
Find and sketch the domain of the functions

$$f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1} \qquad g(x, y) = x \ln(y^2 - x)$$

Solution (Part 1)

The domain of f cut out by $x \neq 1$ and the inequality

$$0 \leq x + y + 1 \iff -x - 1 < y$$

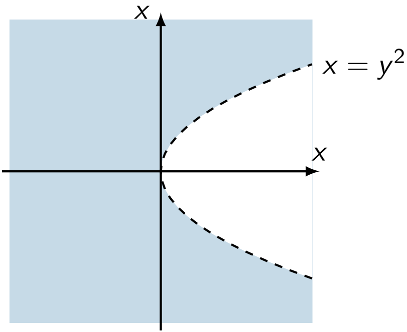


Functions of Two Variables

Solution (Part 2)

The domain is cut out by the inequality

$$0 < y^2 - x \iff x < y^2$$



Functions of Two Variables

Exercise

Sketch the domain of the function

$$f(x,y) = \sqrt{9 - x^2 - y^2}$$

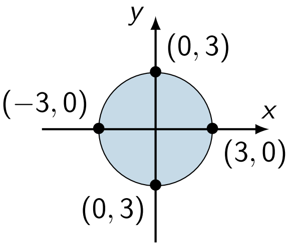
Functions of Two Variables

Solution

The domain is cut out by the inequality

$$0 \le 9 - x^2 - y^2$$

which is the disc of radius 3 about the origin



Graphs

Definition (Graph)

If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

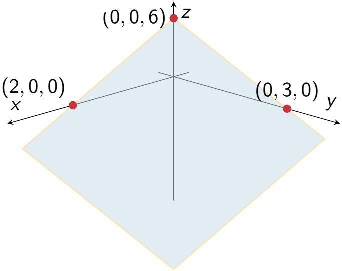
Exercise

Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Graphs

Solution

Write $z = f(x, y) = 6 - 3x - 2y \iff 3x + 2y + z - 6 = 0$ to recognize this as the plane with normal vector $\mathbf{n} = \langle 3, 2, 1 \rangle$ through $(2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 6)$.



Graphs

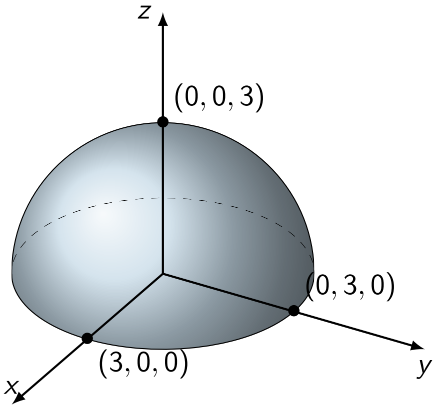
Exercise

Sketch the graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$.

Graphs

Solution

Write $z^2 = f(x, y)^2 = 9 - x^2 - y^2 \iff x^2 + y^2 + z^2 = 9$ to recognize this as the top half of the sphere of radius 3 about the origin



Example

Find the domain and range of $f(x, y) = 4x^2 + y^2$. Use a computer to sketch a graph.

Solution

The domain is $\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$. The range is all real numbers. To visualize this surface, we can use SageMathCell (https://sagecell.sagemath.org).

Graphs

```
# Set variable names
var("x,y,z")
implicit_plot3d(
    z == 4*x^2 + y^2, # Defining equation
    (x,-1,1), # Min/max x
    (y,-1,1), # Min/max y
    (z,0,1)   # Min/max y
)
```

Level Curves and Contour Maps

Definition (Level Curve)

A **level curve** of a function f of two variables is a curve with equation $f(x, y) = k$, where k is a constant in the range of f.

Definition (Contour Map)

A collection of level curves is called a **contour map**.

Level Curves

Exercise

Consider the surface $x^2 + y^2 = z^2 - 1$. Choose any number $a \geq 1$ and consider the intersection of this surface with the plane $z = a$. What do you see?

Level Curves

Solution (Part 1)

- When $a = 1$, the level curve $x^2 + y^2 = 0$ is the degenerate circle $(0, 0)$.

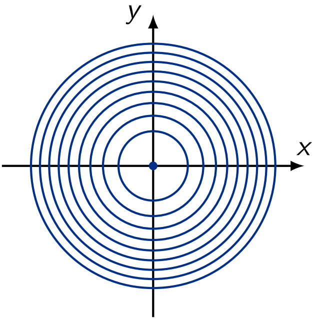
Level Curves

Solution (Part 1)

- When $a = 1$, the level curve $x^2 + y^2 = 0$ is the degenerate circle $(0, 0)$.
- When $a > 1$, the level curve $x^2 + y^2 = a^2 - 1$ is a circle of radius $\sqrt{a^2 - 1}$.

Solution (Part 2)

From the contour map, which is a collection of concentric circles, we surmise that $x^2 + y^2 = z^2 - 1$ is a circular cone for $z \geq 1$. By symmetry, the same is true for $z \leq -1$. This tells us $x^2 + y^2 = z^2 - 1$ is a hyperbola of two sheets.



Functions of More Than Two Variables

Definition (Function of Three Variables)

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

Graphs of Functions of More Than Two Variables

Definition (Level Surface)

A **level surface** of a function of three variables is a surface given by $f(x, y, z) = k$, where k is a constant in the range of f.

Example

The function

$$w = \sqrt{z - x^2 - 2y^2}$$

has the elliptic paraboloid

$$z = x^2 + 2y^2 + a^2$$

as its level surface for each $a \geq 0$.

Graphs of Functions of More Than Two Variables

Definition (Function of n Variables)

A **function of n variables**, f, is a rule that assigns to each ordered n-tupe $(x_1, x_2, \ldots, x_n)$ a unique real number denoted by $f(x_1, x_2, \ldots, x_n)$.