Vector Functions

13.4 Motion in Space: Velocity and Acceleration

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Calculus III

Velocity, Speed, and Acceleration

Definition (Velocity Vector)

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. The **velocity vector** at time t is

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

The **speed** of the particle at time t is the magnitude of the velocity vector, $|\mathbf{v}(t)|$.

Definition (Acceleration)

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. The **acceleration** of the particle is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

Velocity, Speed, and Acceleration

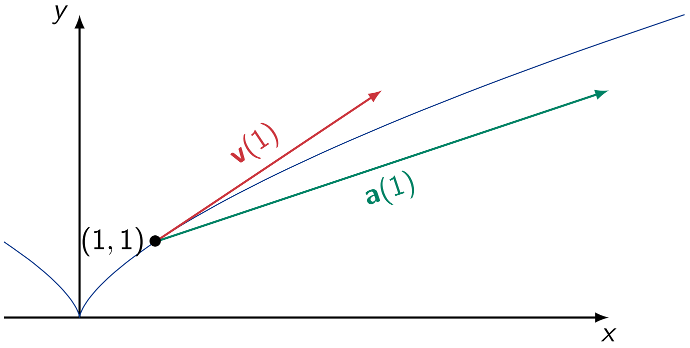
Exercise

The position vector of an object moving in a plane is given by $\mathbf{r}(t) = \langle t^3, t^2 \rangle$. Find its velocity, speed, and acceleration when $t = 1$, and illustrate geometrically.

Velocity, Speed, and Acceleration

Solution

$$\mathbf{v}(1) = \langle 3, 2 \rangle \qquad |\mathbf{v}(1)| = \sqrt{13} \qquad \mathbf{a}(1) = \langle 6, 2 \rangle$$



Velocity, Speed, and Acceleration

Exercise

Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.

Velocity, Speed, and Acceleration

Solution

$$\mathbf{v}(t) = \langle 2t, e^t, (1+t)e^t \rangle$$

$$|\mathbf{v}(t)| = \sqrt{4t^2 + e^{2t} + (1+t)^2 e^{2t}}$$

$$\mathbf{a}(t) = \langle 2, e^t, (2+t)e^t \rangle$$

Exercise

A moving particle starts at an initial position $\mathbf{r}(0) = \langle 1, 0, 0 \rangle$ with initial velocity $\mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$. Its acceleration is $\mathbf{a}(t) = 4t\mathbf{i} + 6t\mathbf{j} + \mathbf{k}$. Find its velocity and position at time t.

Velocity, Speed, and Acceleration

Solution

$$\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \langle 4t, 6t, 1 \rangle \, dt = \langle 2t^2 + 1, 3t^2 - 1, t + 1 \rangle$$

$$\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{v}(t) = \left\langle \frac{2}{3}t^3 + t + 1, t^3 - t, \frac{1}{2}t^2 + t \right\rangle$$

Velocity, Speed, and Acceleration

Newton's Second Law of Motion

If at any time t, a force $\mathbf{F}(t)$ acts on an object of mass m producing an acceleration $\mathbf{a}(t)$, then

$$\mathbf{F}(t) = m\mathbf{a}(t)$$

Velocity, Speed, and Acceleration

Exercise

An object with mass m that moves in a circular path with constant angular speed ω has position vector $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$. Find the force acting on the object and show that it is directed toward the origin.

Velocity, Speed, and Acceleration

Solution

Observe the force is directed toward the origin if it acts in the opposite direction of the position vector (i.e. there is a scalar c such that $\mathbf{F} = -c\mathbf{r}$. Compute

$$\mathbf{v}(t) = a\omega\langle -\sin(\omega t), \cos(\omega t)\rangle \quad \mathbf{a}(t) = -a\omega^2\langle \cos(\omega t), \sin(\omega t)\rangle$$
$$= -\omega^2 \mathbf{r}(t)$$

$$\mathbf{F}(t) = -m\omega^2\mathbf{r}(t)$$

Velocity, Speed, and Acceleration

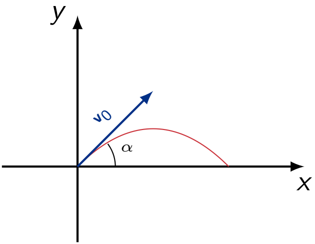
Remark

A force such as the one in the previous example is called a **centripetal** (center-seeking) force.

Projectile Motion

Exercise

A projectile is fired with an angle of elevation α and initial velocity $\mathbf{v}_0$. Assuming air resistance is negligible and the only external force is due to gravity, find the position function $\mathbf{r}(t)$ of the projectile. What value of α maximizes the range (the horizontal distance travelled)?



Projectile Motion

Solution (Part 1)

Let $g \approx 9.8\text{m}/\text{s}^2$ be the magnitude of the force due to gravity.

$$\mathbf{v}(t) = \mathbf{v}_0 + \int_0^t \langle 0, -g \rangle \, dt = \langle 0, -gt \rangle + \mathbf{v}_0$$

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \langle 0, -gt \rangle \, dt + \int_0^t \mathbf{v}_0 \, dt = \left\langle 0, -\frac{1}{2}gt^2 \right\rangle + \mathbf{v}_0 t \\ &= \left\langle |\mathbf{v}_0| \cos(\alpha) t, -t \left(\frac{1}{2}gt - |\mathbf{v}_0| \sin(\alpha) \right) \right\rangle\end{aligned}$$

Projectile Motion

Solution (Part 2)

Notice the y-coordinate is zero at $t_0 = 0$ and

$$t_f = \frac{2\,|\mathbf{v}_0|}{g}\sin(\alpha).$$

Hence the range of the projectile is

$$|\mathbf{v}_0|\cos(\alpha)t_f = \frac{|\mathbf{v}_0|^2}{g}2\cos(\alpha)\sin(\alpha) = \frac{|\mathbf{v}_0|^2}{g}\sin(2\alpha)$$

Projectile Motion

Solution (Part 3)

We see the range is maximized by $\alpha = \pi/4$ using the second derivative test. In particular,

$$\frac{d}{dt}\frac{2\left|\mathbf{v}_0\right|^2}{g}\sin(2\alpha) = \frac{2\left|\mathbf{v}_0\right|^2}{g}\cos(2\alpha) = 0 \iff \alpha = \frac{\pi}{4}$$

$$\frac{d^2}{dt^2}\frac{2\left|\mathbf{v}_0\right|^2}{g}\sin(2\alpha)\Big|_{\alpha=\frac{\pi}{4}} = -\frac{4\left|\mathbf{v}_0\right|^2}{g}\sin\left(\frac{\pi}{2}\right) = -\frac{4\left|\mathbf{v}_0\right|^2}{g} < 0$$

Projectile Motion

Exercise

A projectile is fired with initial speed 150 m/s and angle of elevation 30° from a position 10 m above ground level. Where does the projectile hit the ground, and with what speed?

Projectile Motion

Solution (Part 1)

Modify the previous position vector with $\mathbf{r}(0) = \langle 0, 10 \rangle$ and $\mathbf{v}_0 = \langle 75\sqrt{3}, 75 \rangle$ to obtain

$$\mathbf{v}(t) = \langle 75\sqrt{3}, -gt + 75 \rangle$$

$$\mathbf{r}(t) = \left\langle 75\sqrt{3}t, -\frac{1}{2}gt^2 + 75t + 10 \right\rangle$$

Projectile Motion

Solution (Part 2)

The projectile hits the ground at

$$t_f = \frac{-75 - \sqrt{5821}}{-g} = \frac{75 + \sqrt{5821}}{g} \approx 15.4383141062\text{s}$$

with velocity

$$|\mathbf{v}(t_f)| = \left|\langle 75\sqrt{3}, -\sqrt{5821}\rangle\right| \approx 150.651916682\text{m/s}$$