Vector Functions

13.3 Arc Length and Curvature

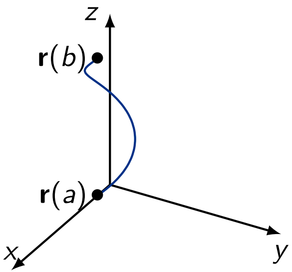
B. Farman

Mathematics and Statistics
Louisiana Tech University

Calculus III

Arc Length

Suppose we have a differentiable vector-valued function, $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, and we would like to measure the length of the space curve over the interval $[a, b]$.

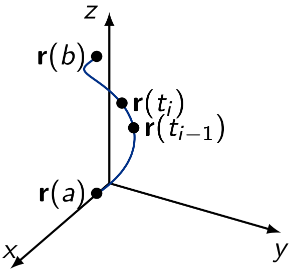


Arc Length

We start by partitioning the interval $[a, b]$ into n subintervals using the points

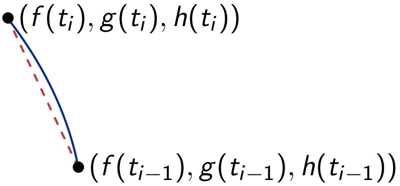
$$t_0 = a < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

and considering the segment of the curve over the i^{th} subinterval $[t_{i-1}, t_i]$.



Arc Length

Since we do not know how to measure the length of the space curve on the subinterval $[t_{i-1}, t_i]$, we replace it with a line segment. The simplest line segment to choose is the one that connects the points $\mathbf{r}(t_{i-1})$ and $\mathbf{r}(t_i)$. However, we need to be slightly more clever to make things work nicely.



Arc Length

Instead of using the endpoint directly, we use linear approximations to its coordinates.

$$
\begin{aligned}
f(t_i) &\approx f'(t_{i-1})(t_i - t_{i-1}) + f(t_{i-1}) = f'(t_{i-1})\Delta t + f(t_{i-1}) \\
g(t_i) &\approx g'(t_{i-1})(t_i - t_{i-1}) + g(t_{i-1}) = g'(t_{i-1})\Delta t + g(t_{i-1}) \\
h(t_i) &\approx h'(t_{i-1})(t_i - t_{i-1}) + h(t_{i-1}) = h'(t_{i-1})\Delta t + h(t_{i-1})
\end{aligned}
$$

Arc Length

Using the linear approximation, the difference between the coordinates is

$$f'(t_i)\Delta t + f(t_{i-1}) - f(t_{i-1}) = f'(t_{i-1})\Delta t$$
$$g'(t_i)\Delta t + g(t_{i-1}) - g(t_{i-1}) = g'(t_{i-1})\Delta t$$
$$h'(t_i)\Delta t + h(t_{i-1}) - h(t_{i-1}) = h'(t_{i-1})\Delta t$$

Arc Length

Hence the length of the line segment that approximates the space curve over $[t_{i-1}, t_i]$ is

$$
\begin{aligned}
L_i &= \sqrt{\left(f'(t_{i-1})\Delta t\right)^2 + \left(g'(t_{i-1})\Delta t\right)^2 + \left(h'(t_{i-1})\Delta t\right)^2} \\
&= \sqrt{\left[f'(t_{i-1})^2 + g'(t_{i-1})^2 + h'(t_{i-1})^2\right] \Delta t^2} \\
&= \sqrt{f'(t_{i-1})^2 + g'(t_{i-1})^2 + h'(t_{i-1})^2}\Delta t = \left|\mathbf{r}'(t_{i-1})\right| \Delta t
\end{aligned}
$$

We approximate the length of the space curve $\mathbf{r}$ over $[a, b]$ by $\sum_{i=1}^{n} L_i$.

Arc Length

Arc Length

The length, L, of the space curve $\mathbf{r}(t)$ over the interval $[a, b]$ is

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \left|\mathbf{r}'(t)\right| \Delta t = \int_a^b \left|\mathbf{r}'(t)\right| dt$$

Arc Length

Remark

While we focused on the construction for functions $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$, there is nothing special about 3-dimensional space. In general, if $\mathbf{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t)\rangle$, then the length of the curve C in $\mathbb{R}^n$ over $[a, b]$ is

$$\int_a^b \left|r'(t)\right| dt$$

Arc Length

Exercise

Find the length of the arc of the circular helix

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$$

from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

Arc Length

Solution

$$\int_0^{2\pi} \sqrt{\sin^2(t) + \cos^2(t) + 1^2}\, dt = \sqrt{2} \int_0^{2\pi} dt$$
$$= 2\sqrt{2}\pi.$$

Arc Length

Remark

A curve space curve has different parameterizations, but the arc length is independent of the parameterization.

Arc Length

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Exercise

The helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ could also be parameterized as $\mathbf{s}(t) = \langle \cos(2t), \sin(2t), 2t \rangle$, $0 \leq t \leq \pi$. It is easy to check $\int_0^{\pi} |\mathbf{s}'(t)| \, dt = 2\sqrt{2}\pi$.

The Arc Length Function

Definition (Arc Length Function)

Suppose C is a space curve given by $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ on $[a, b]$, where $\mathbf{r}'$ is continuous and C is traversed *exactly* once on $[a, b]$. The **arc length function** is

$$s(t) = \int_a^t \left|\mathbf{r}'(u)\right| du$$

As a consequence of part 1 of the Fundamental Theorem of calculus, $s'(t) = |\mathbf{r}'(t)|$

The Arc Length Function

In some instances, it is possible to rewrite the parameter t in terms of the arc length, s. Being able to **parameterize a curve with respect to arc length** is sometimes valuable because the arc length is invariant of the parameterization. For instance, if we can rewrite $\mathbf{r}$ in terms of s, then $\mathbf{r}(s)$ represents the position vector s units along the curve.

The Arc Length Function

Exercise

Re parameterize the helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ with respect to arc length measured from $(1, 0, 0)$ in the direction of increasing t.

The Arc Length Function

Solution

We already know $|\mathbf{r}'(t)| = \sqrt{2}$, so

$$s = \sqrt{2}\int_0^t du = \sqrt{2}t \iff t = \frac{s}{\sqrt{2}}$$

and thus

$$\mathbf{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \right\rangle$$

Curvature

Definition (Smooth)

A parameterization, $\mathbf{r}(t)$, is called **smooth** on an interval I if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on I. A curve is called **smooth** if it has a smooth parameterization.

Curvature

Definition (Curvature)

Let C be a smooth curve. The curvature of C,

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/\,dt}{ds/\,dt}\right| = \left|\frac{\mathbf{T}'(t)}{\mathbf{r}'(T)}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

measures how quickly the curve changes direction at a point.

Curvature

Exercise

Show the curvature of a circle of radius a is $1/a$.

Curvature

Solution

Take the parameterization $\mathbf{r}(t) = \langle a\cos(t), a\sin(t)\rangle$. Compute

$$\begin{aligned}
\mathbf{r}'(t) &= \langle -a\sin(t), \cos(t)\rangle \\
\mathbf{T}(t) &= \langle -\sin(t), \cos(t)\rangle \\
\mathbf{T}'(t) &= \langle -\cos(t), -\sin(t)\rangle \\
\kappa &= \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}
\end{aligned}$$

Curvature

Theorem

Assume C is a smooth curve parameterized by $\mathbf{r}$. The curvature of C is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Exercise

Find the curvature of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at a generic point and at $(0, 0, 0)$.

Curvature

Solution

Compute

$$
\begin{aligned}
r'(t) &= \langle 1, 2t, 3t^2 \rangle \\
r''(t) &= \langle 0, 2, 6t \rangle \\
r' \times r'' &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle
\end{aligned}
$$

Curvature

Solution (Part 2)

$$\left|r' \times r''\right| = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$\left|r'\right|^3 = \sqrt{9t^4 + 4t^2 + 1}^3$$

$$\kappa(t) = 2\sqrt{\frac{9t^4 + 9t^2 + 1}{(9t^4 + 4t^2 + 1)^3}}$$

$$\kappa(0) = 2$$

Curvature

Theorem

Given a smooth plane curve $y = f(x)$*, the parameterization* $\mathbf{r}(x) = \langle x, f(x) \rangle$ *yields*

$$\kappa(x) = \frac{|f''(x)|}{[1 + f'(x)^2]^{3/2}}$$

Exercise

Find the curvature of the parabola $y = x^2$ at the points $(0, 0)$, $(1, 1)$, and $(2, 4)$.

Curvature

Solution

Compute

$$f'(x) = 2x \qquad f''(x) = 2 \qquad \kappa(x) = \frac{2}{(1+4x^2)^{3/2}}$$

$$\kappa(0) = 2 \qquad \kappa(1) = \frac{2}{25}\sqrt{5} \qquad \kappa(2) = \frac{2}{289}\sqrt{17}$$

The Normal and Binormal Vectors

Definition (Principal Unit Normal Vector)

Let C be a smooth curve parameterized by $\mathbf{r}$. For any point where $\kappa \neq 0$, the **principal unit normal vector** (or **unit normal**) is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

and indicates the direction in which the curve is turning at this point.

The Normal and Binormal Vectors

Definition (Binormal Vector)

For a smooth curve, C, the **binormal vector**

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

is a unit vector perpendicular to both $\mathbf{T}$ and $\mathbf{N}$.

The Normal and Binormal Vectors

Remark

The three orthogonal unit vectors **T**, **N**, and **B** provide what is known as the **TNB frame**. These vectors form a basis for $\mathbb{R}^3$, similar to **i**, **j**, and **k**.

The Normal and Binormal Vectors

Exercise

Find the unit normal and binormal vectors for the circular helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

The Normal and Binormal Vectors

Solution

$$\mathbf{r}'(t) = \langle -\sin(t), \cos(t), 1\rangle$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 0\rangle$$

$$\mathbf{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle$$

The Normal and Binormal Vectors

Solution

$$\mathbf{B}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{\sin(t)}{\sqrt{2}} & \frac{\cos(t)}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\cos(t)}{\sqrt{2}} & -\frac{\sin(t)}{\sqrt{2}} & 0 \end{vmatrix}$$

$$= \frac{\sqrt{2}}{2} \langle \sin(t), -\cos(t), 1 \rangle$$

The Normal and Binormal Vectors

Exercise

Find the unit tangent, unit normal, and binormal vectors and the curvature for $\mathbf{r}(t) = \langle t, \sqrt{2}\ln(t), 1/t \rangle$.

The Normal and Binormal Vectors

Solution (Part 1)

$$\mathbf{r}'(t) = \frac{1}{t^2}\left\langle t^2, \sqrt{2}t, -1\right\rangle$$

$$\left|\mathbf{r}'(t)\right| = \frac{t^2+1}{t^2}$$

$$\mathbf{T}(t) = \frac{1}{t^2+1}\left\langle t^2, \sqrt{2}t, -1\right\rangle$$

$$\mathbf{T}'(t) = -\frac{2t}{(t^2+1)^2}\langle t^2, \sqrt{2}t, -1\rangle + \frac{1}{t^2+1}\left\langle 2t, \sqrt{2}, 0\right\rangle$$

The Normal and Binormal Vectors

Solution (Part 2)

$$\mathbf{T}(1) = \frac{1}{2}\langle 1, \sqrt{2}, -1\rangle$$

$$\mathbf{T}'(1) = \frac{1}{2}\langle 1, 0, 1\rangle$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}\langle 1, 0, 1\rangle$$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \frac{1}{2}\langle 1, -\sqrt{2}, -1\rangle$$

The Normal and Binormal Vectors

Solution (Part 3)

$$
\begin{aligned}
\kappa(1) &= \frac{|\mathbf{T}(1)|}{|\mathbf{r}(1)|} \\
&= \frac{\sqrt{22}}{2} \\
&= \frac{\sqrt{2}}{4}
\end{aligned}
$$

The Normal and Binormal Vectors

Definition (Normal Plane)

The plane determined by the vectors **N** and **B** and a point P on the curve C is called the **normal plane** of C at P.

The Normal and Binormal Vectors

Definition (Osculating Plane)

The plane determined by the vectors $\mathbf{T}$ and $\mathbf{N}$ and a point P on the curve C is called the **osculating plane** of C at P.

Definition (Circle of Curvature)

The **circle of curvature** or **osculating circle** of C at P is the circle in the osculating plane that passes through P with radius $1/\kappa$ and center adistance $1/\kappa$ from P along the vector $\mathbf{N}$. The center of the circle is called the **center of curvature** of C at P.

The Normal and Binormal Vectors

Exercise

Find the equations of the normal plane and osculating plane of the helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$ at $(0, 1, \pi/2)$.

The Normal and Binormal Vectors

Solution (Part 1)

When $t = \pi/2$

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\langle -1, 0, 1\rangle$$

$$\mathbf{N}\left(\frac{\pi}{2}\right) = \langle 0, -1, 0\rangle$$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{\sqrt{2}}{2}\langle 1, 0, 1\rangle$$

The Normal and Binormal Vectors

Solution (Part 2)

For the normal plane, use the normal vector $\sqrt{2}\mathbf{T}(\pi/2) = \langle -1, 0, 1 \rangle$ to obtain the equation of the plane

$$\begin{aligned} 0 &= \left\langle -1, 0, 1 \right\rangle \cdot \left\langle x, y-1, z-\frac{\pi}{2} \right\rangle \\ &= -x + z - \frac{\pi}{2} \end{aligned}$$

The Normal and Binormal Vectors

Solution (Part 3)

For the osculating plane, use the normal vector $\sqrt{2}\mathbf{B}(\pi/2) = \langle 1, 0, 1\rangle$ to obtain the equation of the plane

$$\begin{aligned} 0 &= \left\langle 1, 0, 1\right\rangle \cdot \left\langle x, y-1, z-\frac{\pi}{2}\right\rangle \\ &= x + z - \frac{\pi}{2} \end{aligned}$$

Exercise

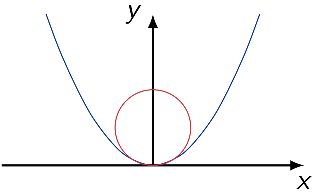
Find and graph the osculating circle $y = x^2$ at the origin.

The Normal and Binormal Vectors

Solution

At the origin, $\mathbf{T}(0) = \langle 1, 0 \rangle$, $\mathbf{N}(0) = \langle 0, 1 \rangle$, and $\kappa = 2$. Hence the osculating circle is

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$$



Torsion

Definition (Torsion)

The **torsion** of a curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{d\mathbf{B}/\,dt}{ds/\,dt} \cdot \mathbf{N} = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|}$$

Torsion

Theorem

The torsion of the curve given by the vector-valued function $\mathbf{r}$ *is*

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

Torsion

Exercise

Find the torsion of the helix $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$.

Torsion

Solution

$$\left|\mathbf{r}'(t)\right| = \sqrt{2} \qquad \mathbf{N}(t) = \langle -\cos(t), -\sin(t), 0\rangle$$

$$\mathbf{B}'(t) = \frac{1}{\sqrt{2}}\langle \cos(t), \sin(t), 0\rangle \qquad \mathbf{B}'(t)\cdot\mathbf{N} = -\frac{1}{\sqrt{2}}$$

$$\tau = -\frac{-1/\sqrt{2}}{\sqrt{2}} = \frac{1}{2}$$