Vector Functions

13.2 Vector Functions and Space Curves

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Calculus III

Derivatives

Definition (Derivative)

The **derivative** of a vector-valued function is

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

provided the limit exists.

Derivatives

Definition (Tangent Vector)

If $\mathbf{r}(t)$ is a differentiable vector-valued function, then $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided $\mathbf{r}'(t) \neq \mathbf{0}$.

Derivatives

Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, *where* f, g, *and* h *are differentiable functions, then*

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Derivatives

Definition (Unit Tangent Vector)

A unit vector that has the same direction as the tangent vector is called the **unit tangent vector**, $\mathbf{T}$, and is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Derivatives

Exercise

1. Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin(2t)\mathbf{k}$.
2. Find the unit tangent vector at the point where $t = 0$.

Derivatives

Solution (Part 1)

The derivative is

$$\mathbf{r}'(t) = \left\langle 3t^2, (1-t)e^{-t}, 2\cos(2t) \right\rangle$$

Derivatives

Solution (Part 2)

The tangent vector at $t = 0$ is $\mathbf{r}'(0) = \langle 0, 1, 2 \rangle$, which has magnitude $\sqrt{0^2 + 1^2 + 2^2} = \sqrt{5}$. Therefore the unit tangent vector is

$$\mathbf{T}(t) = \left\langle 0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right\rangle$$

Derivatives

Exercise

Let $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2 - t)\mathbf{j}$. Find $\mathbf{r}'(t)$. Sketch $\mathbf{r}(1)$ and $\mathbf{r}'(1)$.

Derivatives

Solution (Part 1)

Compute

$$\mathbf{r}'(t) = \left\langle \frac{1}{2\sqrt{t}}, -1 \right\rangle$$

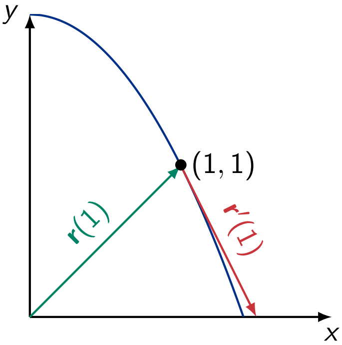
$$r(1) = \langle 1, 1 \rangle$$

$$r'(1) = \left\langle \frac{1}{2}, -1 \right\rangle$$

Derivatives

Solution (Part 2)

Now sketch the vectors:



Derivatives

Exercise

Find parametric equations for the line tangent to the helix

$$x = 2\cos(t) \qquad y = \sin(t) \qquad z = t$$

at $(0, 1, \pi/2)$.

Derivatives

Solution (Part 1)

The tangent line is

$$\mathbf{r}'\left(\frac{\pi}{2}\right)t + \left\langle 0, 1, \frac{\pi}{2}\right\rangle = \left\langle -2\sin\left(\frac{\pi}{2}\right)t, \cos\left(\frac{\pi}{2}\right)t + 1, t + \frac{\pi}{2}\right\rangle$$
$$= \left\langle -2t, 1, t + \frac{\pi}{2}\right\rangle$$

Differentiation Rules

Theorem

Suppose $\mathbf{u}$ *and* $\mathbf{v}$ *are differentiable,* c *is a scalar, and* f *is a real-valued function.*

1. $\frac{d}{dt}(\mathbf{u}+\mathbf{v})(t) = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}c\mathbf{u}(t) = c\mathbf{u}'(t)$
3. $\frac{d}{dt}(f\mathbf{u})(t) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})(t) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\frac{d}{dt}(\mathbf{u}\times\mathbf{v})(t) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
6. $\frac{d}{dt}(\mathbf{u}\circ f)(t) = f'(t)\mathbf{u}'(f(t))$

Differentiation Rules

Exercise

Assume $|\mathbf{r}(t)| = c$ is constant. Show for all t, $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Solution

Write $c^2 = |\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$ and apply part (4) of the theorem above to see

$$\begin{aligned} 0 = \frac{d}{dt}(c^2) &= \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) \\ &= 2\mathbf{r}'(t) \cdot \mathbf{r}(t) \end{aligned}$$

implies $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$.

Integrals

Definition (Definite Integral)

The **definite integral** of a continuous vector-valued function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ on $[a, b]$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle$$

Integrals

Theorem (Fundamental Theorem of Calculus)

If $\mathbf{R}$ *is an anti-derivative for* $\mathbf{r}$ *on* $[a, b]$ *(i.e. for all* $a \leq t \leq b$*,* $\mathbf{R}'(t) = \mathbf{r}(t)$*), then*

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a)$$

Integrals

Exercise

Find an anti-derivative or $\mathbf{r}(t) = \langle 2\cos(t), \sin(t), 2t \rangle$, and use your answer to evaluate

$$\int_0^{\frac{\pi}{2}} \mathbf{r}(t)\, dt$$

Integrals

Solution

The anti-derivatives for $\mathbf{r}$ are

$$\mathbf{R}(t) = \langle 2\sin(t), -\cos(t), t^2 \rangle + \mathbf{C}$$

where $\mathbf{C}$ is any (constant) vector. To evaluate the definite integral, take $\mathbf{C} = \mathbf{0}$ and compute

$$\mathbf{R}\left(\frac{\pi}{2}\right) - \mathbf{R}(0) = \left\langle 2, 0, \frac{\pi^2}{4} \right\rangle - \langle 0, -1, 0 \rangle = \left\langle 2, 1, \frac{\pi^2}{4} \right\rangle$$