Vector Functions

13.1 Vector Functions and Space Curves

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Calculus III

Vector Functions and Space Curves

Definition (Vector-valued Function)

A **vector-valued function** is a function whose domain is a set of real numbers and whose range is a set of vectors,

$$\mathbf{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t) \rangle$$

The real-valued functions $f_1, f_2, \ldots, f_n$ are called the **component functions** of $\mathbf{r}$.

Example

The function

$$\mathbf{r}(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$$

has component functions

$$f(t) = t^3 \qquad g(t) = \ln(3-t) \qquad h(t) = \sqrt{t}$$

The domain of $\mathbf{r}$ is $[0, 3)$.

Limits and Continuity

Definition (Limit)

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

provided the limits of the component functions exist.

Limits and Continuity

Exercise

Let $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\mathbf{j} + \sin(t)/t\mathbf{k}$. Find $\lim_{t \to 0} \mathbf{r}(t)$.

Limits and Continuity

Solution

$$\lim_{t\to 0}\left\langle (1+t^3), te^{-t}, \frac{\sin(t)}{t}\right\rangle = \left\langle \lim_{t\to 0}(1+t^3), \lim_{t\to 0} te^{-t}, \lim_{t\to 0}\frac{\sin(t)}{t}\right\rangle$$
$$= \langle 1, 0, 1\rangle$$

Limits and Continuity

Definition (Continuous)

A vector function $\mathbf{r}$ is **continuous at** a if $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

Space Curves

Definition (Space Curve)

Suppose that f, g, and h are continuous, real-valued functions on an interval I. The set C of all points (x, y, z) in space, where

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and t varies throughout the interval I is called a **space curve**. The equations above are called the **parametric equations of** C, and t is called a **parameter**.

Space Curves

Exercise

Describe the curve defined by

$$\mathbf{r}(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$$

Solution

Write

$$\mathbf{r}(t) = \langle 1, 5, 6 \rangle t + \langle 1, 2, -1 \rangle$$

to see this is the line through the point $(1, 2, -1)$ parallel to the vector $\langle 1, 5, 6 \rangle$.

Space Curves

Exercise

Sketch

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$$

Space Curves

Solution

This is a helical path along the cylinder of radius 1.

Exercise

Find a vector equation for the line segment joining the points $(1, 3, -2)$ and $(2, -1, 3)$.

Space Curves

Solution

Take the direction vector

$$\mathbf{v} = \langle 2-1, -1-3, 3+2 \rangle = \langle 1, -4, 5 \rangle$$

and the initial vector $\mathbf{v}_0 = \langle 1, 3, -2 \rangle$ to obtain

$$\mathbf{r}(t) = t\mathbf{v} + \mathbf{v}_0 = \langle t+1, -4t+3, 5t-2 \rangle$$

Exercise

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

Solution

Since the intersection lies on the cylinder of radius 1, we can write $x = \cos(t)$, $y = \sin(t)$ and $z = 2 - \sin(t)$ or

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 2 - \sin(t) \rangle$$

on the interval $0 \leq t \leq 2\pi$.

Exercise

Find parametric equations for the intersection of $4y = x^2 + z^2$ and $y = x$.

Solution

Write

$$\begin{aligned} 4x = x^2 + z^2 &\iff x^2 - 4x + 4 + z^2 = 4 \\ &\iff (x-2)^2 + z^2 = 2^2 \end{aligned}$$

to recognize this as a circle of radius 2 in the plane $y = x$

$$\mathbf{r}(t) = \langle 2 + 2\cos(t), 2 + 2\cos(t), 2\sin(t) \rangle, \; 0 \le t \le 2\pi$$

Using Technology to Draw Space Curves

Example

Use the `parametric_plot3d` function in SageMath (sagecell.sagemath.org) to draw the **twisted cubic**

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$$

Hint: Type `parametric_plot3d?` for help with syntax.

Using Technology to Draw Space Curves