Vectors and the Geometry of Space

12.6 Cylinders and Quadric Surfaces

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Lines

Definition (Cylinder)

A **cylinder** is a surface that consists of all lines (called **rulings**) that are parallel to a given line and and pass through a given plane curve.

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Remark

Intuitively, this says the surface is built from shifted copies of a plane curve.

Lines

Example

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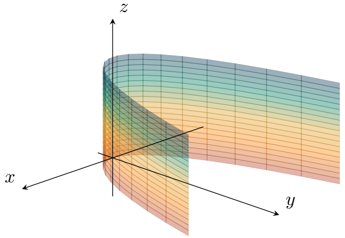
- At every point on the parabola, add a perpendicular line (i.e. a line parallel to the z-axis).
- The effect is that of stacking infinitely many parabolas along the z-axis called **traces**.
- The result, called a **parabolic cylinder**, looks like a slightly bent sheet of paper.



Other parabolic cylinders are constructed similarly.

Example

1. The solutions to $z = 2x^2$ forms a parabolic cylinder along the y-axis (as opposed to the z-axis). Each trace is a parabola in the xz-plane that has been stretched by a factor of 2.

Lines

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2. The solutions to $y = z^2 + 2z + 1$ forms a parabolic cylinder along the x-axis. Each trace is a copy of the parabola $y = (z+2)^2$ in the yz-plane.

Any conic section (parabola, ellipse, hyperbola, etc.) can also be used to construct a cylinder. The familiar right cylinder is constructed from circles.

Example

The cylinder

$$(x-h)^2 + (y-k)^2 = r^2$$

is a hollow tube of radius r extending parallel to the z-axis. For each value of z, the trace is a circle of radius r centered about the point (h, k, z).

Quadric Surfaces

Definition

A **quadric surface** is the graph of an equation of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

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Remark

It is possible to rewrite any quadric surface in one of two standard forms

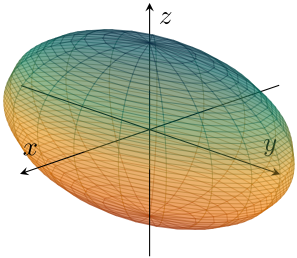
$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

Ellipsoid

An **ellipsoid** has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses.



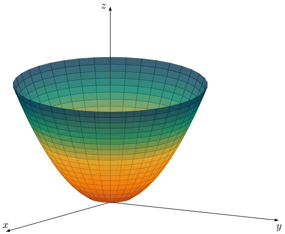
Quadric Surfaces

Elliptic Paraboloid

An elliptic paraboloid has the form

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Horizontal traces are ellipses.
- Vertical traces are parabolas.



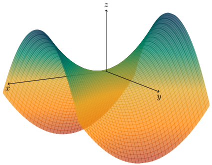
Quadric Surfaces

Hyperbolic Paraboloid

Hyperbolic paraboloids have the form

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

- Horizontal traces are hyperbolas.
- Vertical traces are parabolas.

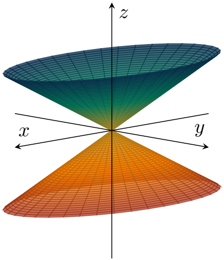


Quadric Surfaces

Cone

Cones have the form
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

- Horizontal traces are ellipses.
- Vertical traces in $x = k$ and $y = k$, $k \neq 0$ are hyperbolas.
- Vertical trace in $x = 0$ and $y = 0$ is a pair of lines.



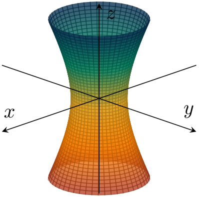
Quadric Surfaces

Hyperboloid of One Sheet

Hyperboloids of one sheet have the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

- Horizontal traces are ellipses.
- Vertical traces are hyperbolas.



Quadric Surfaces

Hyperboloid of Two Sheets

Hyperboloids of two sheets have the form

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

- Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$.
- Vertical traces are hyperbolas.