Vectors and the Geometry of Space

12.5 Equations of Lines and Planes

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Calculus III

Lines

Vector Equation of a Line

Assume we are given two points P and Q. Let $\mathbf{r}_0 = \vec{OP}$ and $\mathbf{v} = \vec{QP}$. The vector equation of the **line** through P and Q is

$$\mathbf{r}(t) = t\mathbf{v} + \mathbf{r}_0$$

Lines

Parametric Equations of a Line in Space

Assume we are given two points $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$. Write

$$a = x_1 - x_1 \qquad b = y_1 - y_0 \qquad c = z_1 - z_0$$

The **parametric equations** of the through P and Q with parameter t are

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

Lines

Parametric Equations of a Line in Space

Assume we are given a point $P = (x_0, y_0, z_0)$ and a direction vector, $\mathbf{v} = \langle a, b, c \rangle$. The **parametric equations** of the through P parallel to $\mathbf{v}$ is

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

Exercise

Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Find two other points on the line.

Lines

Solution

The vector equation of the line is

$$\mathbf{r}(t) = \langle 5, 1, 3\rangle + t\langle 1, 4, -2\rangle t = \langle t+5, 4t+1, -2t+3\rangle$$

and the parametric equations are

$$x = t + 5 \qquad y = 4t + 1 \qquad z = -2t + 3$$

Two other points on the line are $(6, 5, 1)$ and $(7, 9, -1)$.

Lines

Symmetric Equations of a Line in Space

Assume we are given parametric equations

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

for a line in space. If $a \neq 0$, $b \neq 0$, and $c \neq 0$, then solving each equation for t yields the **symmetric equations** of the line

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Exercise

Find parametric and symmetric equations for the line through $(2, 4, -3)$ and $(3, -1, 1)$. Where does this line intersect the xy-plane?

Lines

Solution (Part 1)

The parametric equations are

$$x = t + 2 \qquad y = -5t + 4 \qquad z = 4t - 3$$

and the symmetric equations are

$$x - 2 = -\frac{y-4}{5} = \frac{z+3}{4}$$

Lines

Solution (Part 2)

The line intersects the xy-plane when $z = 0$, so the symmetric equations yield

$$x - 2 = \frac{3}{4} \iff x = 2 + \frac{3}{4} = \frac{11}{4}$$

$$-\frac{y-4}{5} = \frac{3}{4} \iff y = -5\left(\frac{3}{4}\right) + 4 = \frac{1}{4}$$

Therefore the line intersects the xy-plane at $(11/4, 1/4, 0)$.

Lines

Line Segment

Assume we are given two points P and Q. Let $\mathbf{r}_0 = \vec{OP}$ and $\mathbf{r}_1 = \vec{PQ}$. The **line segment** that begins at P and ends at Q is given by the vector equation

$$\mathbf{r}(t) = \mathbf{r}_1 t + (1 - t)\mathbf{r}_0$$

with parameter $0 \leq t \leq 1$.

Lines

Definition (Skew Lines)

Two lines in space are called **skew lines** if they are not parallel and they do not intersect.

Lines

Exercise

Show the lines L_1 and L_2 are skew lines.

$$L_1 : x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2 : x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

Lines

Solution (Part 1)

To see these lines are not parallel, we must show the direction vectors $\mathbf{v} = \langle 1, 3, -1 \rangle$ and $\mathbf{w} = \langle 2, 1, 4 \rangle$ for L_1 and L_2 are not parallel. That is, there is no solution to $\mathbf{v} = c\mathbf{w}$. It suffices to observe the equations

$$1 = 2c \qquad 3 = c \qquad -1 = 4c$$

have no solution because, for example, $c = 3$ implies $2c = 6 \neq 1$.

Lines

Solution (Part 2)

To see L_1 and L_2 do not intersect, we must show the equations

$$1 + t = 2s \qquad -2 + 3t = 3 + s \qquad 4 - t = -3 + 4s$$

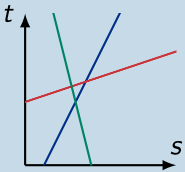
do not have simultaneous solutions. First, rewrite

$$t = 2s - 1 \qquad t = \frac{s}{3} + \frac{5}{3} \qquad t = -4s + 7$$

Lines

Solution (Part 3)

Geometrically, the fact that L_1 and L_2 do not intersect in space is equivalent to observing these three lines do not intersect at a common point in the *st*-plane:



$t = 2s - 1$

$t = \frac{s}{3} + \frac{5}{3}$

$t = -4s + 7$

Solution (Part 4)

Algebraically, the lines $t = 2s - 1$ and $t = s/3 + 5/3$ have a unique solution:

$$2s - 1 = \frac{s}{3} + \frac{5}{3} \iff 6s - 3 = s + 5 \iff 5s = 8 \iff s = \frac{8}{5}$$

$$t = 2\left(\frac{8}{5}\right) - 1 = \frac{11}{5}$$

Lines

Solution (Part 5)

However, $s = 8/5$ and $t = 11/5$ is not on the line $t = -4s + 7$ because

$$-4\left(\frac{8}{5}\right) + 7 = \frac{-32 + 35}{5} = \frac{3}{5} \neq \frac{8}{5}$$

Therefore these lines do not intersect.

Planes

Definition (Plane)

Given a fixed point P_0 and a nonzero vector $\mathbf{n}$, the set of points P in $\mathbb{R}^3$ for which $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$ is called the **plane** through P_0 with **normal vector n**.

Planes

Vector Equation of a Plane

Given a point $P_0 = \langle x_0, y_0, z_0 \rangle$ and normal vector $\mathbf{n} = \langle a, b, c \rangle$, let $\mathbf{r}_0 = \vec{OP}$ and $\mathbf{r} = \langle x, y, z \rangle$. The **vector equation** of the plane through P_0 with normal vector $\mathbf{n}$ is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Planes

Scalar Equation of a Plane

Given a point $P_0 = \langle x_0, y_0, z_0 \rangle$ and normal vector $\mathbf{n} = \langle a, b, c \rangle$, the **scalar equation** of the plane through P_0 with normal vector $\mathbf{n}$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Planes

Exercise

Find an equation of the plane through the point $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$. Find the intercepts and sketch the plane.

Planes

Solution (Part 1)

The plane through $(2, 4, -1)$ with normal vector $\mathbf{n} = \langle 2, 3, 4 \rangle$ is

$$\begin{aligned} 0 &= \langle 2, 3, 4 \rangle \cdot \langle x - 2, y - 4, z + 1 \rangle \\ &= 2(x - 2) + 3(y - 4) + 4(z + 1) \\ &= 2x + 3y + 4z - 12 \end{aligned}$$

or, equivalently, $2x + 3y + 4z = 12$.

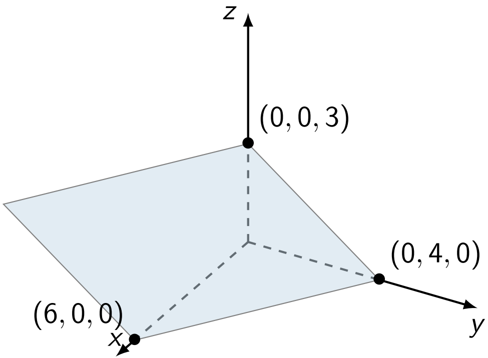
Solution (Part 2)

The intercepts are given by setting two of the coordinates to zero:

$$\begin{aligned} 2x + 3(0) + 4(0) = 12 &\iff x = 6 \\ 2(0) + 3y + 4(0) = 12 &\iff y = 4 \\ 2(0) + 3(0) + 4z = 12 &\iff z = 3 \end{aligned}$$

This tells us the plane intersects the axes at $(6, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 3)$.

Planes



Planes

Exercise

Find an equation of the plane that passes through the points $P = (1, 3, 2)$, $Q = (3, -1, 6)$, and $R = (5, 2, 0)$.

Solution (Part 1)

First compute a normal vector

$$\begin{aligned}
\mathbf{n} &= \vec{PR} \times \vec{PQ} = \langle 4, -1, -2\rangle \times \langle 2, -4, 4\rangle \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & -2 \\ 2 & -4 & 4 \end{vmatrix} = (-4-8)\vec{i} - (16+4)\mathbf{j} + (-16+2)\mathbf{k} \\
&= \langle -12, -20, -14\rangle = -2\langle 6, 10, 7\rangle
\end{aligned}$$

Planes

Solution (Part 2)

Use a normal vector and a point to compute an equation of the plane:

$$\begin{aligned} 0 &= \langle 6, 10, 7 \rangle \cdot \langle x - 1, y - 3, z - 2 \rangle \\ &= 6x - 6 + 10y - 30 + 7z - 14 \\ &= 6x + 10y + 7z - 50 \end{aligned}$$

or, equivalently, $6x + 10y + 7z = 50$.

Planes

Exercise

Find the point at which the line with parametric equations $x = 2 + 3t$, $y = -4t$, $z = 5 + t$ intersects the plane $4x + 5y - 2z = 18$.

Planes

Solution

The point of intersection occurs when

$$\begin{aligned} 18 &= 4(2+3t)+5(-4t)-2(5+t) \\ &= -10t-2 \\ \iff 10t &= -20 \iff t=-2 \end{aligned}$$

corresponding to the point

$$(2+3(-2), -4(-2), 5+(-2)) = (-4, 8, 3).$$

Planes

Definition (Parallel Planes)

We say two planes are parallel if their normal vectors are parallel.

Planes

Definition (Angle Between Planes)

The **angle** between two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ is defined to be the angle between the normal vectors,

$$\theta = \arccos\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$$

Planes

Exercise

Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$, then find the symmetric equations for the line of intersection between them.

Planes

Solution (Part 1)

The normal vectors for these planes are $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, -2, 3 \rangle$, so the angle between the planes is

$$\arccos\left(\frac{1-2+3}{\sqrt{3}\sqrt{14}}\right) = \arccos\left(\frac{2}{\sqrt{42}}\right) \approx 72.0247161901911^\circ$$

Solution (Part 2)

Since the line lies in both planes, it is parallel to the vector

$$\begin{aligned}
\langle 1,1,1\rangle \times \langle 1,-2,3\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} \\
&= (3+2)\mathbf{i} - (3-1)\mathbf{j} + (-2-1)\mathbf{k} \\
&= \langle 5,-2,-3\rangle
\end{aligned}$$

Planes

Solution (Part 3)

It is easy to see the point $(1, 0, 0)$ lies on each plane because

$$1 + 0 + 0 = 1 = 1 - 2(0) + 3(0)$$

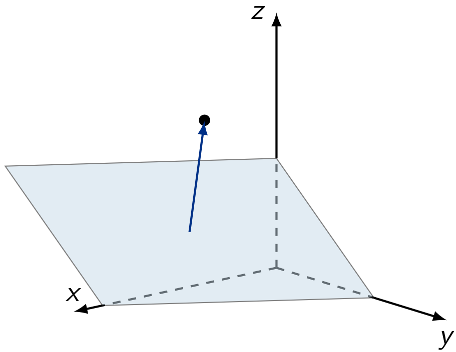
so the symmetric equations for line through $(1, 0, 0)$ parallel to $\langle 5, -2, -3 \rangle$ are

$$\frac{x-1}{5} = -\frac{y}{2} = -\frac{z}{3}$$

Distances

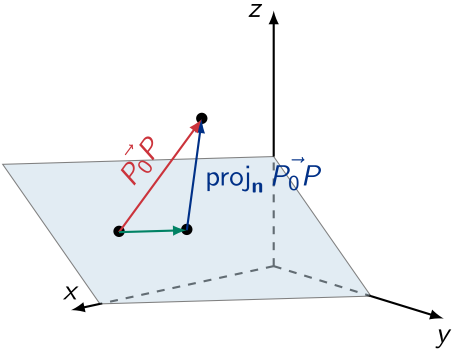
Definition (Distance Between a Point and a Plane)

The **distance between a point** P **and a plane** $ax + by + cz + d = 0$ is the magnitude of an orthogonal vector from the plane to the point.



Distances

Suppose we are given a point $P = (x_1, y_1, z_1)$ and a plane $ax + by + cz + d = 0$. Let $P_0 = (x_0, y_0, z_0)$ be any point in the plane. The orthogonal projection of $\vec{P_0P}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$ creates an orthogonal vector from the plane to P.



Distances

The length of the orthogonal projection of $\vec{P_0P}$ onto $\mathbf{n}$ is then

$$\begin{aligned}
\left|\operatorname{comp}_{\mathbf{n}} \vec{P_0P}\right| &= \frac{\left|\mathbf{n} \cdot \vec{P_0P}\right|}{|\mathbf{n}|} \\
&= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|ax_1 + by_1 + cz_1 + (-ax_0 - by_0 - cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}
\end{aligned}$$

Distances

Theorem

The distance between the point $P = (x, y, z)$ *and the plane* $ax + by + cz + d = 0$ *is*

$$\frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Exercise

Find the distance between the parallel planes $10x + 2y - 2z = 5$ and $5x + y - z = 1$.

Distances

Solution

The point $(0, 1, 0)$ lies on the plane $5x + y - 1 = 1$, so we can compute the distance between the planes via point $(0, 1, 0)$ and the normal vector $\mathbf{n} = \langle 10, 2, -2 \rangle$

$$\frac{|10(0) + 2(1) + (-2)(0) - 5|}{\sqrt{10^2 + 2^2 + (-2)^2}} = \frac{3}{6\sqrt{3}}$$
$$= \frac{\sqrt{3}}{6}$$

Distances

Exercise

Find the distance between the skew lines

$$L_1\colon x = 1 + t \qquad y = -2 + 3t \qquad z = 4 - t$$
$$L_2\colon x = 2s \qquad y = 3 + s \qquad z = -3 + 4s$$

Solution (Part 1)

Write $\mathbf{v} = \langle 1, 3, -1\rangle$, $\mathbf{v}_0 = \langle 1, -2, 4\rangle$, $\mathbf{w} = \langle 2, 1, 4\rangle$, and $\mathbf{w}_0 = \langle 0, 3, -3\rangle$ so that $L_1 \colon \mathbf{v}t + \mathbf{v}_0$ and $L_2 \colon \mathbf{w}t + \mathbf{w}_0$. The normal vector

$$\mathbf{n} = \mathbf{v} \times \mathbf{w} = \langle 13, -6, -5\rangle$$

defines the plane $13x - 6y - 5z = 0$ through the origin containing $\mathbf{v}$ and $\mathbf{w}$.

Distances

Solution (Part 2)

Shifting this plane to pass through $\mathbf{v}_0$ and $\mathbf{w}_0$, respectively, we obtain the parallel planes

$$\mathbf{n} \cdot (\langle x, y, z \rangle - \mathbf{v}_0) = 13x - 6y - 5z - 5 = 0, \text{ and}$$
$$\mathbf{n} \cdot (\langle x, y, z \rangle - \mathbf{w}_0) = 13x - 6y - 5z + 3 = 0$$

that contain L_1 and L_2, respectively. Hence the distance between L_1 and L_2 is the distance between these planes.

Solution (Part 3)

Using the point $\mathbf{v}_0$, the distance between these parallel planes – and hence the distance between L_1 and L_2 – is

$$\frac{|13(1)+(-6)(-2)+(-5)(4)+3|}{\sqrt{13^2+(-6)^2+5^2}} = \frac{8}{\sqrt{230}}$$