Vectors and the Geometry of Space

12.4 The Cross Product

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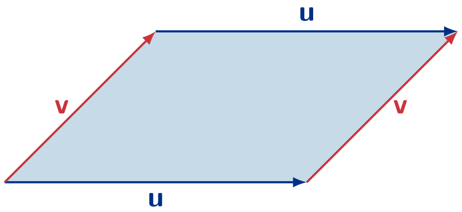
Calculus III

The Cross Product of Two Vectors

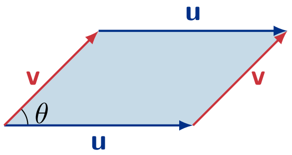
Definition (Cross Product)

The **cross product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is a vector $\mathbf{u} \times \mathbf{v}$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, and has magnitude equal to area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.



The Cross Product of Two Vectors

We can see the area of the parallelogram is $|\mathbf{u}|\,|\mathbf{v}|\sin(\theta)$ using the construction below.



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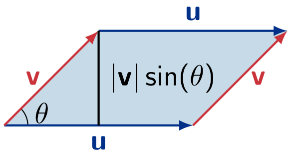

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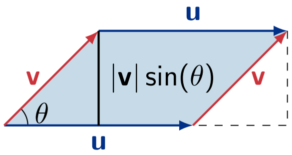

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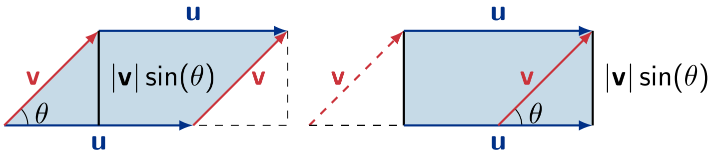


The Cross Product of Two Vectors

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *then*

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$$
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin(\theta)$$

The Cross Product of Two Vectors

Corollary

For vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *the following are equivalent.*

1. *The vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel.*
2. *The angle,* θ, *measured from* $\mathbf{u}$ *to* $\mathbf{v}$ *is either* 0 *or* π,
3. *The cross product of* $\mathbf{u}$ *and* $\mathbf{v}$ *has zero magnitude,* $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin(\theta) = 0$.
4. *The cross product of* $\mathbf{u}$ *and* $\mathbf{v}$ *is the zero vector,* $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

The Cross Product of Two Vectors

Definition (2 × 2 Determinant)

Assume $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$. The **determinant** associated to $\mathbf{u}$ and $\mathbf{v}$ is the signed area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

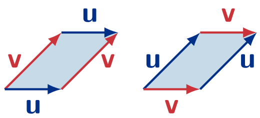


Figure: Positive determinant (left) and negative determinant (right).

The Cross Product of Two Vectors

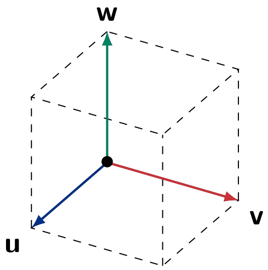
Theorem

The determinant associated to $\langle u_1, u_2 \rangle$ *and* $\langle v_1, v_2 \rangle$ *is*

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$$

Definition (3×3 Determinant)

The determinant associated to the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is the signed volume of the parallelepiped formed by these three vectors.



The Cross Product of Two Vectors

Theorem

The determinant associated to the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is computed recursively by*

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

The Cross Product of Two Vectors

Theorem

The cross product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *can be computed using the symbolic* 3×3 *determinant*

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

The Cross Product of Two Vectors

Exercise

Compute $\langle 1, 3, 4 \rangle \times \langle 2, 7, -5 \rangle$.

The Cross Product of Two Vectors

Solution

$$\begin{aligned}
\langle 1,3,4\rangle \times \langle 2,7,-5\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} \\
&= (-15-28)\mathbf{i} - (-5-8)\mathbf{j} + (7-6)\mathbf{k} \\
&= \langle -43, 13, 1\rangle
\end{aligned}$$

The Cross Product of Two Vectors

Exercise

Find a vector perpendicular to the plane that contains the points $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

The Cross Product of Two Vectors

Solution

If the plane contains the points, then it must contain vectors between any pair of points. Form two vectors, say,

$$\vec{PQ} = \langle -3, 1, -7 \rangle \quad \text{and} \quad \vec{QR} = \langle 3, -6, 2 \rangle$$

then take the cross product

$$\vec{PQ} \times \vec{QR} = \langle -40, -15, 15 \rangle$$

The Cross Product of Two Vectors

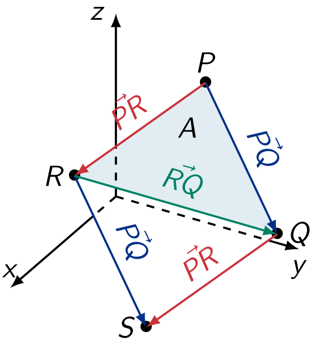
Exercise

Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

Solution

Observe the parallelogram formed by the vectors $\vec{PR}$ and $\vec{PQ}$ is twice the size of the triangle, so the area, A, of the desired triangle is

$$A = \frac{1}{2}\left|\vec{PQ} \times \vec{PR}\right| = \frac{5\sqrt{82}}{2} \approx 22.6384628453435$$



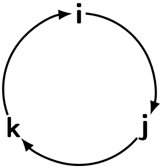
Properties of the Cross Product

Theorem

The standard basis vectors $\mathbf{i}$, $\mathbf{j}$, *and* $\mathbf{k}$ *satisfy*

$$\mathbf{i}\times\mathbf{j}=\mathbf{k} \qquad \mathbf{j}\times\mathbf{k}=\mathbf{i} \qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

$$\mathbf{j}\times\mathbf{i}=-\mathbf{k} \qquad \mathbf{k}\times\mathbf{j}=-\mathbf{i} \qquad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$



Properties of the Cross Product

Theorem

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors and* c *is a scalar, then*

1. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
4. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
5. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

Triple Products

Definition

The **scalar triple product** of the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ is precisely the determinant associated to these vectors

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Triple Products

Definition (Coplanar)

The vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are **coplanar** if they all lie in the same plane.

Triple Products

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Theorem

The vectors $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are coplanar if and only if* $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

Triple Products

Exercise

Show the vectors $\mathbf{u} = \langle 1, 4, -7 \rangle$, $\mathbf{v} = \langle 2, -1, 4 \rangle$, and $\mathbf{w} = \langle 0, -9, 18 \rangle$ are coplanar.

Triple Products

Solution

$$\begin{vmatrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} = (-18+36) - 4(36-0) - 7(-18-0)$$

$$= 18 - 144 + 126$$

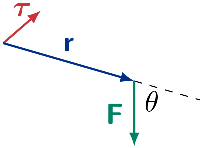
$$= 0$$

Application: Torque

Definition (Torque)

Consider a force $\mathbf{F}$ acting on a rigid body at a point given by a position vector $\mathbf{r}$. The **torque** relative to the origin is

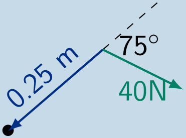
$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$



Application: Torque

Exercise

A bolt is tightened by applying a 40-N force to a 0.25-m wrench at a 75° angle as depicted below. Find the magnitude of the torque about the center of the bolt.



Application: Torque

Solution

The magnitude of the torque is

$$\begin{aligned} |\boldsymbol{\tau}| = |\mathbf{r} \times \mathbf{F}| &= |\mathbf{r}|\,|\mathbf{F}| \sin(\theta) \\ &= 10 \sin\left(\frac{5\pi}{12}\right) \approx 9.65925826289068\,\mathrm{N} \cdot \mathrm{m} \end{aligned}$$