Vectors and the Geometry of Space

12.3 The Dot Product

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Calculus III

The Dot Product of Two Vectors

Definition (The Dot Product)

Assume $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ are vectors. The **dot product** (or **scalar product** or **inner product**) of $\mathbf{u}$ and $\mathbf{v}$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

The Dot Product of Two Vectors

Exercise

Compute the following:

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle$$

$$\langle -1, 7, 4 \rangle \cdot \left\langle 6, 2, -\frac{1}{2} \right\rangle$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k})$$

The Dot Product of Two Vectors

Solution

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 6 - 4 = 2$$

$$\langle -1, 7, 4 \rangle \cdot \left\langle 6, 2, -\frac{1}{2} \right\rangle = -6 + 14 - 2 = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 0 + 4 + 3 = 7$$

The Dot Product of Two Vectors

Theorem (Properties of the Dot Product)

If $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors and* c *is a scalar, then*

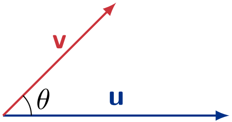
1. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
2. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$
5. $\mathbf{0} \cdot \mathbf{u} = 0$

The Dot Product of Two Vectors

Theorem

Assume $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors. If* θ *is the angle measured from* $\mathbf{u}$ *to* $\mathbf{v}$*, then*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos(\theta) \iff \cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$$



The Dot Product of Two Vectors

Exercise

Assume $\mathbf{u}$ and $\mathbf{v}$ are vectors,

- $\mathbf{u}$ has magnitude 4,
- $\mathbf{v}$ has magnitude 6, and
- the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/3$.

Find $\mathbf{u} \cdot \mathbf{v}$.

The Dot Product of Two Vectors

Solution

$$\mathbf{u} \cdot \mathbf{v} = 24 \cos\left(\frac{\pi}{3}\right) = 24\left(\frac{1}{2}\right) = 12.$$

The Dot Product of Two Vectors

Exercise

Find the angle between $\mathbf{u} = \langle 2, 2, -1 \rangle$ and $\mathbf{v} = \langle 5, -3, 2 \rangle$.

The Dot Product of Two Vectors

Solution

$$\theta = \arccos\left(\frac{2}{\sqrt{9}\sqrt{38}}\right) \approx 1.46 \text{ radians}$$

Definition (Orthogonal)

We say the vectors **u** and **v** are **orthogonal** (or **perpendicular**) if the angle between them is $\pi/2$.

The Dot Product of Two Vectors

Definition (Orthogonal)

We say the vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** (or **perpendicular**) if the angle between them is $\pi/2$.

Theorem

The vectors $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal if and only if*

$$0 = \cos\left(\frac{\pi}{2}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|\,|\mathbf{u}|} \iff 0 = \mathbf{u}\cdot\mathbf{v}$$

The Dot Product of Two Vectors

Exercise

Verify that $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is perpendicular to $5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

The Dot Product of Two Vectors

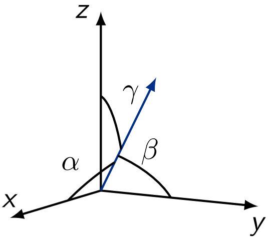
Solution

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 10 - 20 + 10 = 0$$

Direction Angles and Direction Cosines

Definition (Direction Angles)

Assume $\mathbf{v} = \langle a, b, c \rangle$ is a non-zero, three-dimensional vector. The **direction angles**, α, β, and γ, are the angles that $\mathbf{v}$ makes with the positive x-axis, y-axis, and z-axis, respectively.



Direction Angles and Direction Cosines

Definition (Direction Cosines)

Assume $\mathbf{v} = \langle a, b, c \rangle$ is a non-zero, three-dimensional vector. The **direction cosines** of $\mathbf{v}$ are

$$\cos(\alpha) = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|\,|\mathbf{i}|} = \frac{a}{|\mathbf{v}|} \qquad \cos(\beta) = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|\,|\mathbf{j}|} = \frac{b}{|\mathbf{v}|} \qquad \cos(\gamma) = \frac{\mathbf{v}\cdot\mathbf{i}}{|\mathbf{v}|\,|\mathbf{k}|} = \frac{c}{|\mathbf{v}|}$$

Direction Angles and Direction Cosines

Corollary

Assume $\mathbf{v} = \langle a, b, c \rangle$ *is a non-zero, three-dimensional vector. The vector*

$$\begin{aligned} \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle &= \left\langle \frac{a}{|\mathbf{v}|}, \frac{b}{|\mathbf{v}|}, \frac{c}{|\mathbf{v}|} \right\rangle \\ &= \frac{1}{|\mathbf{v}|}\mathbf{v} \end{aligned}$$

is the unit vector in the direction of $\mathbf{v}$.

Direction Angles and Direction Cosines

Exercise

Find the direction angles of the vector $\mathbf{v} = \langle 1, 2, 3 \rangle$.

Direction Angles and Direction Cosines

Solution

$$\alpha = \operatorname{arccos}\left(\frac{1}{\sqrt{14}}\right) \approx 74.4986404331^\circ$$

$$\beta = \operatorname{arccos}\left(\frac{2}{\sqrt{14}}\right) \approx 57.6884667626^\circ$$

$$\gamma = \operatorname{arccos}\left(\frac{3}{\sqrt{14}}\right) \approx 36.6992252005^\circ$$

Projections

Definition

Assume $\mathbf{u}$ and $\mathbf{v}$ are vectors.

- The **orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$** is the vector, $\text{proj}_{\mathbf{v}}\, \mathbf{u}$, parallel to $\mathbf{v}$ that is closest to $\mathbf{u}$.

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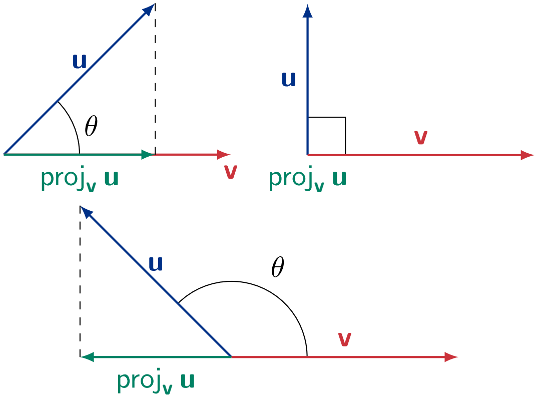
- The **orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$** is the vector, $\text{proj}_{\mathbf{v}}\,\mathbf{u}$, parallel to $\mathbf{v}$ that is closest to $\mathbf{u}$.
- The **scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$**, $\text{comp}_{\mathbf{v}}\,\mathbf{u}$, is the *signed* magnitude of $\text{proj}_{\mathbf{v}}\,\mathbf{u}$.

Projections

There are three possible arrangements of **u** and **v** with $\text{comp}_{\mathbf{v}}\,\mathbf{u}$ positive, zero, and negative, respectively.



Projections

Theorem (Projection Formulas)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors, then*

- $\text{comp}_{\mathbf{v}}\,\mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$

Projections

Theorem (Projection Formulas)

If $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors, then*

- $\operatorname{comp}_{\mathbf{v}} \mathbf{u} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$
- $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = (\operatorname{comp}_{\mathbf{v}} \mathbf{u}) \dfrac{\mathbf{v}}{|\mathbf{v}|} = \left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}$

Projections

Exercise

Find the orthogonal projection and scalar component of $\mathbf{u} = \langle 1, 1, 2 \rangle$ along $\mathbf{v} = \langle -2, 3, 1 \rangle$.

Projections

Solution

$$\text{comp}_{\mathbf{v}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} = \frac{3}{\sqrt{14}}$$

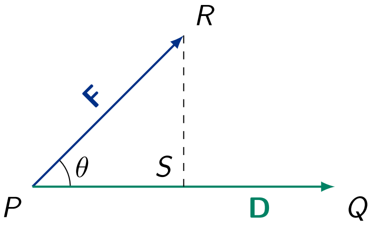
$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \text{comp}_{\mathbf{v}}\,\mathbf{u}\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3}{14}\langle -2, 3, 1\rangle$$

Application: Work

Definition (Work)

Assume a constant force vector, $\mathbf{F} = \vec{PR}$, moves an object from P to Q. The **displacement vector** is $\mathbf{D} = PQ$. If θ is the angle measured from $\mathbf{D}$ to $\mathbf{F}$, then the **work** done by $\mathbf{F}$ is

$$W = (|\mathbf{F}|\cos(\theta))\,|\mathbf{D}| = |\mathbf{F}|\,|\mathbf{D}|\cos(\theta) = \mathbf{F}\cdot\mathbf{D}$$



Exercise

A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle 35° above the horizontal. Find the work done by the force.

Application: Work

Solution

$$W = (|\mathbf{F}|\cos(\theta))\,|\mathbf{D}| = 70\cos\left(\frac{7\pi}{36}\right)100 \approx 5734.06431002\text{ J}$$

Application: Work

Exercise

A force is given by vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $(2, 1, 0)$ to the point $(4, 6, 2)$. Find the work done.

Application: Work

Solution

The work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} \\ &= \langle 3, 4, 5 \rangle \cdot \langle 4 - 2, 6 - 1, 2 - 0 \rangle \\ &= \langle 3, 4, 5 \rangle \cdot \langle 2, 5, 2 \rangle \\ &= 6 + 20 + 10 \\ &= 36 \end{aligned}$$