Vectors and the Geometry of Space

12.2 Vectors

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Calculus III

Geometric Description of Vectors

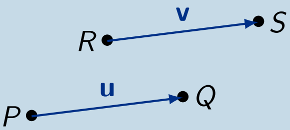
Definition (Vector)

A **vector** is a quantity that has both length (or magnitude) and direction. A vector with its tail at the point P (called the **initial point**) and its head at the point Q (called the **terminal point**) is denoted by $\mathbf{v} = \overrightarrow{PQ}$.

Geometric Description of Vectors

Convention (Equivalent Vectors)

Two vectors, $\mathbf{u}$ and $\mathbf{v}$, are **equivalent** (or **equal**) if they have the same magnitude and direction and we write $\mathbf{u} = \mathbf{v}$.



Geometric Description of Vectors

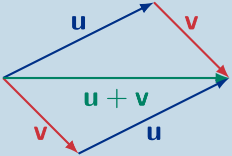
Definition (Zero Vector)

The **zero vector**, $\mathbf{0}$, is the unique vector with length zero and no specific direction.

Geometric Description of Vectors

Definition (Vector Addition)

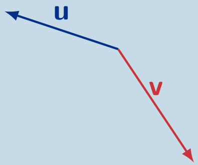
If **u** and **v** are vectors positioned so the tail of **v** is at the head of **u**, then the **sum of u and v** is the vector $\mathbf{u} + \mathbf{v}$ from the tail of **u** to the head of **v**.



Geometric Description of Vectors

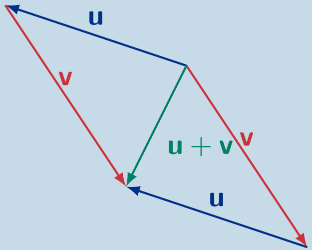
Exercise

Draw the sum of the vectors below.



Geometric Description of Vectors

Solution



Geometric Description of Vectors

Convention (Scalar)

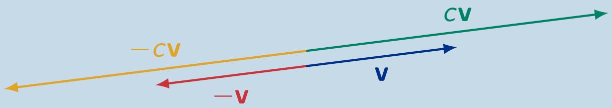
Real numbers that have a magnitude, but not a direction, are called **scalars**.

Geometric Description of Vectors

Definition (Scalar Multiplication)

Assume $c > 0$ is a scalar and $\mathbf{v}$ is a vector.

- $-\mathbf{v}$ is the vector with the opposite direction as $\mathbf{v}$.
- $c\mathbf{v}$ is the vector with the same direction as $\mathbf{v}$ and c units longer.



Geometric Description of Vectors

Definition (Parallel Vector)

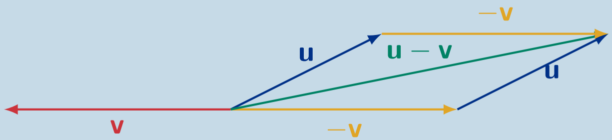
The vector $\mathbf{u}$ is **parallel** to the vector $\mathbf{v}$ if there exists a non-zero scalar, c, such that $\mathbf{u} = c\mathbf{v}$.

Geometric Description of Vectors

Convention (Difference of Vectors)

The **difference** of vectors **u** and **v** is defined to be

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$



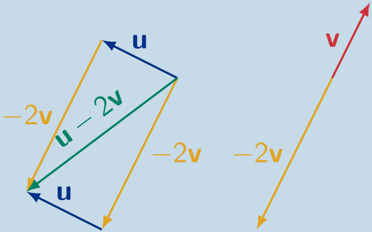
Geometric Description of Vectors

Exercise

Draw $\mathbf{u} - 2\mathbf{v}$.

Geometric Description of Vectors

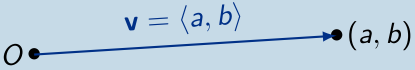
Solution



Components of a Vector

Definition (Two-Dimensional Position Vector)

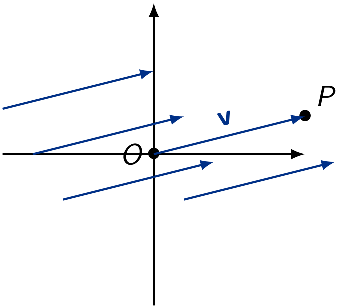
A vector, $\mathbf{v}$, with its tail at the origin, O, and its head at the point (a, b) has x-**component** a and y-**component** b. We call $\mathbf{v} = \langle a, b \rangle$ the **position vector** of the point (a, b).



Components of a Vector

Definition (Two-Dimensional Representation)

We call any geometric vector a **representation** of the algebraic vector $\mathbf{v} = \langle a, b \rangle$ if the two vectors are equivalent.



Components of a Vector

Algebraic Vectors from Points in the Plane

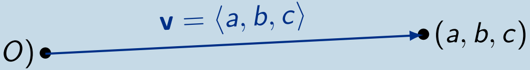
Given points $P(x_0, y_0)$ and $Q(x_1, y_1)$, the vector $\mathbf{v}$ with representation $\vec{PQ}$ is

$$\mathbf{v} = \langle x_1 - x_0, y_1 - y_0 \rangle$$

Components of a Vector

Definition (Three-Dimensional Position Vector)

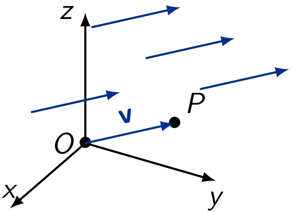
A vector, $\mathbf{v}$, with its tail at the origin, O, and its head at (a, b, c) has x-**component** a, y-**component** b, and z-**component** $z_1 - z_2$. We call $\mathbf{v} = \langle a, b, c \rangle$ the **position vector** of the point (a, b, c).



Components of a Vector

Definition (Three-Dimensional Representation)

We call any geometric vector a **representation** of the algebraic vector $\mathbf{v} = \langle a, b, c \rangle$ if the two vectors are equivalent.



Components of a Vector

Algebraic Vectors from Points in Space

Given points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$, the vector $\mathbf{v}$ with representation $\vec{PQ}$ is

$$\mathbf{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$

Components of a Vector

Exercise

Find the vector represented by the directed line segment with initial point $(2, -3, 4)$ and terminal point $(-2, 1, 1)$.

Components of a Vector

Solution

This is the vector

$$\begin{aligned} \mathbf{v} &= \langle -2-2, 1-(-3), 1-4 \rangle \\ &= \langle -4, 4, -3 \rangle \end{aligned}$$

Components of a Vector

Definition (Magnitude)

The **magnitude** of the vector $\mathbf{v}$ is the length of any of its representation and is denoted by $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

Components of a Vector

Magnitude of a Vector

The magnitude of $\mathbf{v} = \langle a, b \rangle$ is

$$|\mathbf{v}| = \sqrt{a^2 + b^2}.$$

The magnitude of $\mathbf{v} = \langle a, b, c \rangle$ is

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}.$$

Components of a Vector

Algebraic Operations with Position Two-Dimensional Vectors

Assume $\mathbf{u} = \langle x_0, y_0 \rangle$, $\mathbf{v} = \langle x_1, y_1 \rangle$, and c is a scalar. Then

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle x_0 + x_1, y_0 + y_1 \rangle, \\ \mathbf{u} - \mathbf{v} &= \langle x_0 - x_1, y_0 - y_1 \rangle, \text{ and} \\ c\mathbf{u} &= \langle cx_0, cy_0 \rangle. \end{aligned}$$

Components of a Vector

Algebraic Operations with Position Three-Dimensional Vectors

Assume $\mathbf{u} = \langle x_0, y_0, z_0 \rangle$, $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$, and c is a scalar. Then

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= \langle x_0 + x_1, y_0 + y_1, z_0 + z_1 \rangle, \\
\mathbf{u} - \mathbf{v} &= \langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle, \text{ and} \\
c\mathbf{u} &= \langle cx_0, cy_0, cz_0 \rangle.
\end{aligned}$$

Exercise

Let $\mathbf{u} = \langle 4, 0, 3 \rangle$ and $\mathbf{v} = \langle -2, 1, 5 \rangle$. Find $|\mathbf{u}|$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $2\mathbf{u} + 5\mathbf{v}$.

Components of a Vector

Solution

$$
\begin{aligned}
|\mathbf{u}| &= \sqrt{4^2 + 0^2 + 3^2} = \sqrt{25} = 5 \\
\mathbf{u} + \mathbf{v} &= \langle 4 + -2, 0 + 1, 3 + 5 \rangle = \langle 2, 1, 8 \rangle \\
\mathbf{u} - \mathbf{v} &= \langle 4 - (-2)\,, 0 - 1, 3 - 5 \rangle = \langle 6, -1, -2 \rangle \\
2\mathbf{u} + 5\mathbf{v} &= \langle 2(4) + 5(-2), 2(0) + 5(1), 2(3) + 5(5) \rangle \\
&= \langle 8 - 10, 5, 6 + 25 \rangle \\
&= \langle -2, 5, 31 \rangle
\end{aligned}
$$

Components of a Vector

Definition (n-dimensional Vectors)

An n-**dimensional vector** is an ordered n-tuple of scalars,

$$\mathbf{v} = \langle a_1, a_2, \ldots, a_n \rangle.$$

We denote by V_n the set of all n-dimensional vectors – often called a (real) vector space (of dimension n).

Components of a Vector

Theorem (Properties of Vectors)

Assume $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *are vectors in* V_n, *and* c *and* d *are scalars.*

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. $(cd)\mathbf{u} = c(d\mathbf{u})$
8. $1\mathbf{u} = \mathbf{u}$

Components of a Vector

Standard Basis Vectors

The vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

are called the **standard basis vectors**.
To say these are basis vectors means every element of V_3 can be written (uniquely) as a sum of these vectors:

$$\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Components of a Vector

Exercise

Let $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + 7\mathbf{k}$. Express the vector $2\mathbf{u} + 3\mathbf{v}$ in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Components of a Vector

Solution

$$
\begin{aligned}
2\mathbf{u} + 3\mathbf{v} &= 2\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 12\mathbf{i} + 21\mathbf{k} \\
&= 2\mathbf{i} + 12\mathbf{i} + 4\mathbf{j} - 6\mathbf{k} + 21\mathbf{k} \\
&= (2 + 12)\mathbf{i} + 4\mathbf{j} + (-6 + 21)\mathbf{k} \\
&= 14\mathbf{i} + 4\mathbf{j} + 15\mathbf{k}
\end{aligned}
$$

Components of a Vector

Definition

A **unit vector** is a vector with magnitude 1.

Remark

The vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are all unit vectors.

Components of a Vector

Unit Vector in the Direction of a Vector

Assume $\mathbf{v}$ is a vector. The vector

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v}$$

is a unit vector that points in the same direction as $\mathbf{v}$.

Components of a Vector

Exercise

Find the unit vector in the direction of $\mathbf{v} = 2\mathbf{i} - 1\mathbf{j} - 2\mathbf{k}$.

Components of a Vector

Solution

The magnitude of **v** is

$$|\mathbf{v}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

so the unit vector in the direction of **v** is

$$\mathbf{u} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

Applications

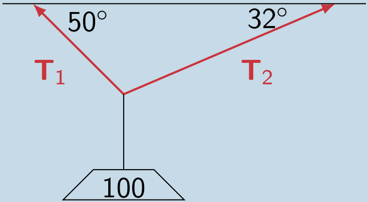
Definition

If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Applications

Exercise

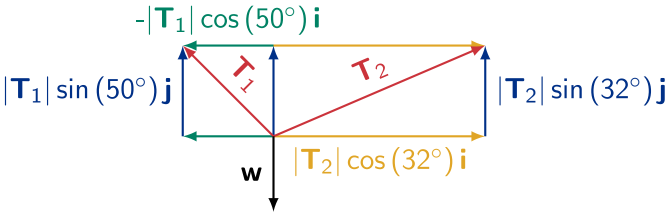
A 100-lb weight hangs from two wires. Find the tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in the wires and the magnitudes of these tensions.



Solution (Part 1)

Model using $\mathbf{w} = -100\mathbf{j}$ and $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\mathbf{j}$. Decompose $\mathbf{T}_1$ and $\mathbf{T}_2$ into the component forces in the x and y directions:

$$0 = -|\mathbf{T}_1|\cos(50^\circ) + |\mathbf{T}_2|\cos(32^\circ)$$
$$100 = |\mathbf{T}_1|\sin(50^\circ) + |\mathbf{T}_2|\sin(32^\circ)$$



Applications

Solution (Part 2)

Solve the equation $0 = -|\mathbf{T}_1|\cos(50^\circ) + |\mathbf{T}_2|\cos(32^\circ)$ for $|\mathbf{T}_1|$ in terms of $|\mathbf{T}_2|$:

$$|\mathbf{T}_1| = \frac{|\mathbf{T}_2|\cos(32^\circ)}{\cos(50^\circ)}$$

Applications

Solution (Part 3)

Solve for $|\mathbf{T}_2|$ using

$$|\mathbf{T}_1| = \frac{|\mathbf{T}_2| \cos(32^\circ)}{\cos(50^\circ)}$$

and $100 = |\mathbf{T}_1| \sin(50^\circ) + |\mathbf{T}_2| \sin(32^\circ)$. This yields

$$|\mathbf{T}_2| = \frac{100}{\tan(50^\circ)\cos(32^\circ) + \sin(32^\circ)}$$

Applications

Solution (Part 4)

Solve for $|\mathbf{T}_1|$ using

$$|\mathbf{T}_2| = \frac{100}{\tan(50^\circ)\cos(32^\circ) + \sin(32^\circ)}$$

and $100 = |\mathbf{T}_1|\sin(50^\circ) + |\mathbf{T}_2|\sin(32^\circ)$. This yields

$$|\mathbf{T}_1| = \frac{100\cos(32^\circ)}{\cos(50^\circ)\left(\tan(50^\circ)\cos(32^\circ) + \sin(32^\circ)\right)}$$

Applications

Solution (Part 5)

Plug the value for $|\mathbf{T}_1|$ into the formula for $\mathbf{T}_1$:

$$\begin{aligned}\mathbf{T}_1 &= -\frac{100\cos(32^\circ)}{\cos(50^\circ)\left(\tan(50^\circ)\cos(32^\circ)+\sin(32^\circ)\right)}\cos(50^\circ)\,\mathbf{i}\\ &\quad+\frac{100\cos(32^\circ)}{\cos(50^\circ)\left(\tan(50^\circ)\cos(32^\circ)+\sin(32^\circ)\right)}\sin(50^\circ)\,\mathbf{j}\\ &\approx -55.047196394\mathbf{i}+65.6026940648\mathbf{j}\end{aligned}$$

Applications

Solution (Part 6)

Plug the value of $|\mathbf{T}_2|$ into the formula for $\mathbf{T}_2$:

$$\begin{aligned}\mathbf{T}_2 &= \frac{100}{\tan(50^\circ)\cos(32^\circ) + \cos(32^\circ)}\cos(50^\circ) \\ &\quad + \frac{100}{\tan(50^\circ)\cos(32^\circ) + \sin(32^\circ)}\sin(50^\circ) \\ &\approx 55.047196394\mathbf{i} + 34.3973059352\mathbf{j}\end{aligned}$$

Applications

Exercise

A woman launches a boat from the south shore of a straight river that flows directly west at 4 mi/h. She wants to land at the point directly across on the opposite shore. If the speed of the boat (relative to the water) is 8 mi/h, in what direction should she steer the boat in order to arrive at the desired landing point?

Solution (Part 1)

Choose variables to model the situation.

- Choose coordinates so the y-axis points north and the x-axis points east.

Applications

Solution (Part 1)

Choose variables to model the situation.

- Choose coordinates so the y-axis points north and the x-axis points east.
- Label current in the river $\mathbf{v}_c = -4\mathbf{i}$.

Applications

Solution (Part 1)

Choose variables to model the situation.

- Choose coordinates so the y-axis points north and the x-axis points east.
- Label current in the river $\mathbf{v}_c = -4\mathbf{i}$.
- Label the angle between the boat and bank θ.

Applications

Solution (Part 1)

Choose variables to model the situation.

- Choose coordinates so the y-axis points north and the x-axis points east.
- Label current in the river $\mathbf{v}_c = -4\mathbf{i}$.
- Label the angle between the boat and bank θ.
- Label the velocity of the boat $\mathbf{v}_b = 8\cos(\theta)\mathbf{i} + 8\sin(\theta)\mathbf{j}$.

Solution (Part 1)

Choose variables to model the situation.

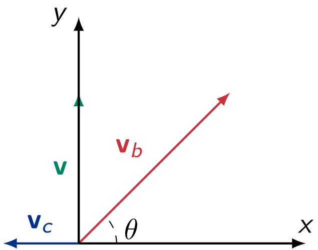
- Choose coordinates so the y-axis points north and the x-axis points east.
- Label current in the river $\mathbf{v}_c = -4\mathbf{i}$.
- Label the angle between the boat and bank θ.
- Label the velocity of the boat $\mathbf{v}_b = 8\cos(\theta)\mathbf{i} + 8\sin(\theta)\mathbf{j}$.
- The track of the boat is defined to be $\mathbf{v} = \mathbf{v}_c + \mathbf{v}_b$.

Applications

Solution (Part 2)

Diagram the boat's path and observe that

$$\mathbf{v} = |\mathbf{v}|\,\mathbf{j} = \mathbf{v}_c + \mathbf{v}_b = (8\cos(\theta) - 4)\mathbf{i} + 8\sin(\theta)\mathbf{j}$$



Applications

Solution (Part 4)

Equating the components of the velocity yields the equations

$$0 = 8\cos(\theta) - 4$$
$$|v| = 8\sin(\theta)$$

Solving the first equation for θ yields

$$\theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$