Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate Systems

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Calculus III

3D Space

Definition (Rectangular Coordinates)

The set

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is called the **three-dimensional rectangular coordinate system**.

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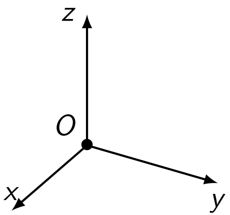
The set

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is called the **three-dimensional rectangular coordinate system**. We call the marked point $(0, 0, 0)$ the **origin**, O.

3D Space

We visualize the three-dimensional rectangular coordinate system as having three perpendicular directions called x, y, and z.



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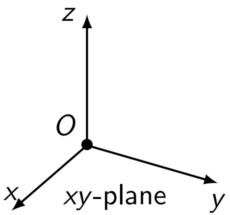

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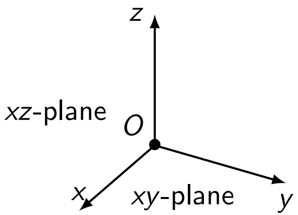

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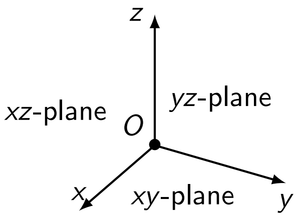

Surfaces and Solids

In two dimensions, a single equation in the variables x and y defines a curve.

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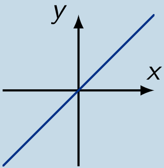
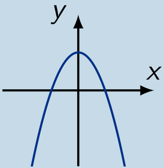
Example

The equations $y - x = 0$ and $x^2 + y = 2$.





Surfaces and Solids

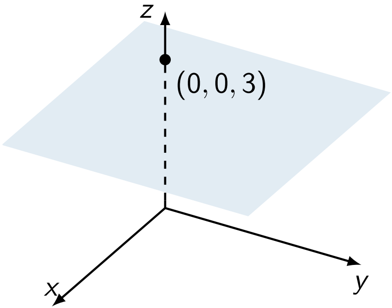
In three dimensions, a single equation in the variables x, y, and z defines a surface.

Exercise

What surface in $\mathbb{R}^3$ is represented by the equation $z = 3$?

Solution

The equation $z = 3$ represents a plane parallel to the xy-plane.



Surfaces and Solids

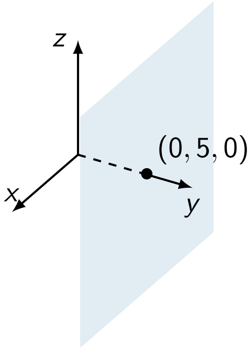
Exercise

What surface in $\mathbb{R}^3$ is represented by the equation $y = 5$?

Surfaces and Solids

Solution

The equation $y = 5$ represents a plane parallel to the xz-plane.



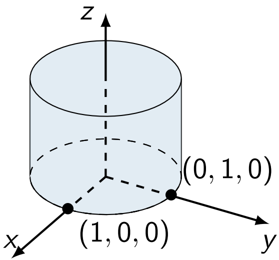
Exercise

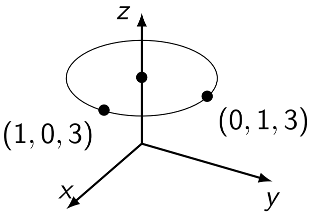
Describe the points in $\mathbb{R}^3$ that satisfy **both**

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 3.$$

Solution

The points that satisfy $x^2 + y^2 = 1$ form a cylinder of radius 1 centered at the z-axis. The points that also satisfy $z = 3$ form a circular cross section of the cylinder, 3 units above the xy-plane.



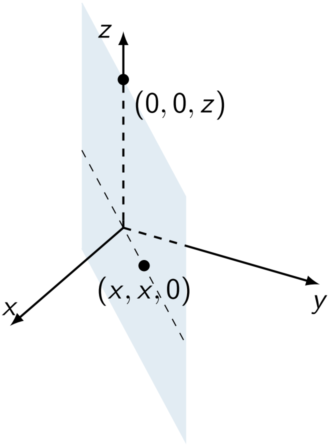


Exercise

Sketch the surface in $\mathbb{R}^3$ represented by the equation $y = x$.

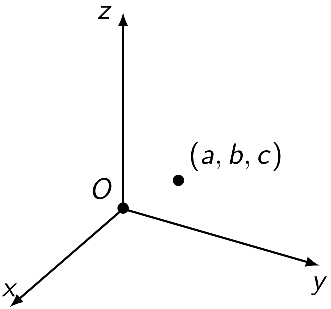
Solution

The equation $y = x$ is a plane that passes through the z-axis at a 45 degree angle.



Distance and Spheres

Suppose we wish to find the distance between the origin, O, and a point P with coordinates (a, b, c).



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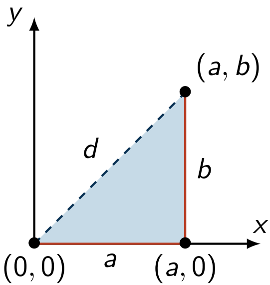
- The distance in the plane between the origin and the point (a, b) is $d = \sqrt{a^2 + b^2}$.
- The distance formula is a consequence of the Pythagorean Theorem.



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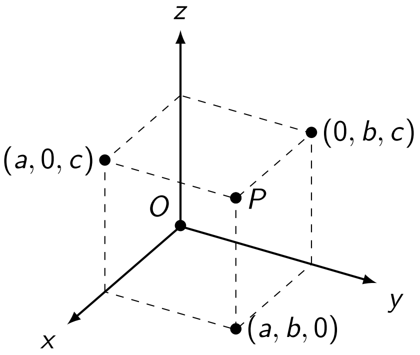
- The **projection of** P **onto the** yz**-plane** is the point $(0, b, c)$.

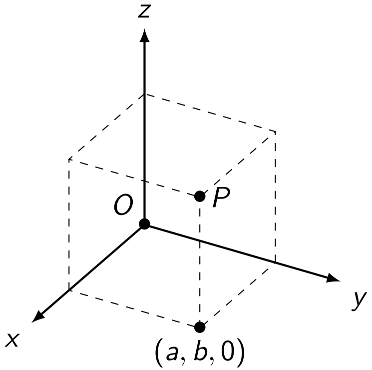
Distance and Spheres

The projections of P on the xy-, xz-, and yz-planes.



Distance and Spheres

We form a right triangle using the origin, the point P, and the projection onto the xy-plane.



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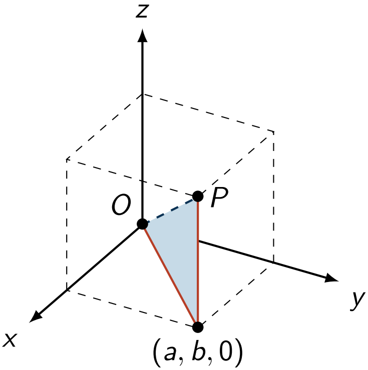

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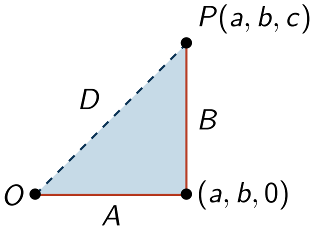

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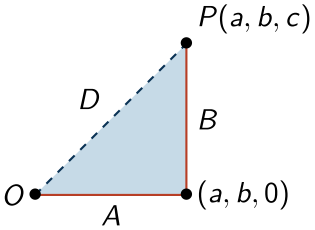

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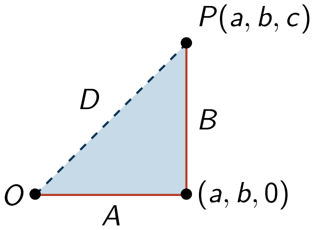

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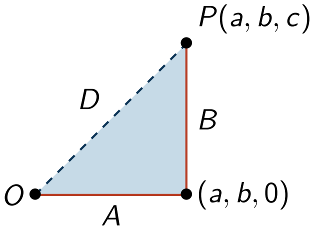

- The length of the bottom is $A = \sqrt{a^2 + b^2}$.
- The length of the side is $B = c$.
- The distance from O to P is

$$D = \sqrt{A^2 + B^2} = \sqrt{a^2 + b^2 + c^2}.$$

Distance and Spheres

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- The distance between two points (x_0, y_0) and (x_1, y_1) in the plane is
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- The distance between two points (x_0, y_0) and (x_1, y_1) in the plane is
$$\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}.$$
- The distance between two points (x_0, y_0, z_0) and (x_1, y_1, z_1) in space is
$$\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

Distance and Spheres

Theorem (Distance Formula in Three Dimensions)

The distance between two points (x_0, y_0, z_0) *and* (x_1, y_1, z_1) *in space is*

$$\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}.$$

Exercise

Find the distance from the point $(2, -1, 7)$ to the point $(1, -3, 5)$.

Distance and Spheres

Solution

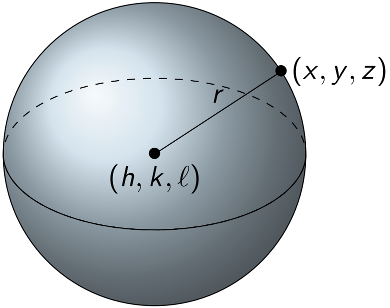
The distance between $(2, -1, 7)$ and $(1, -3, 5)$ is

$$\begin{aligned}\sqrt{(2-1)^2 + (-1-(-3))^2 + (7-5)^2} &= \sqrt{(-1)^2 + 2^2 + 2^2} \\ &= \sqrt{1+4+4} \\ &= \sqrt{9} \\ &= 3\end{aligned}$$

Distance and Spheres

Definition (Sphere)

The sphere of radius r about (h, k, ℓ) consists of the points in $\mathbb{R}^3$ that are *exactly* r units from (h, k, ℓ).



Theorem (Equation of a Sphere)

The sphere of radius r centered at (h, k, ℓ) consists of the solutions to the equation

$$(x-h)^2+(y-k)^2+(z-\ell)^2=r^2$$

Exercise

Find an equation of the sphere with center $(3, -1, 6)$ that passes through $(5, 2, 3)$.

Solution

Using the Distance Formula in Three Dimensions and the Equation of a Sphere,

$$\begin{aligned}(x-3)^2+(y+1)^2+(z-6)^2 &= (3-5)^2+(-1-2)^2+(6-3)^2\\ &= 2^2+(-3)^2+3^2\\ &= 4+9+9\\ &= 22\end{aligned}$$

Distance and Spheres

Exercise

Show that

$$x^2 + y^2 + z^2 + 4x - 6y + 2z + 6 = 0$$

represents a sphere. Find its center and radius.

Distance and Spheres

Solution

Complete the square three times:

$$\begin{aligned} x^2 + y^2 + z^2 + 4x - 6y + 2z &= -6 \\ \left(x^2 + 4x + 4\right) + \left(y^2 - 6y + 9\right) + \left(z^2 + 2z + 1\right) &= -6 + 4 + 9 + 1 \\ (x+2)^2 + (y-3)^2 + (z+1)^2 &= 8 \end{aligned}$$

The center is $(-2, 3, -1)$ and the radius is $\sqrt{8} = 2\sqrt{2}$.

Exercise

What region in $\mathbb{R}^3$ satisfies

$$1 \leq x^2 + y^2 + z^2 \leq 4 \quad \text{and} \quad z \leq 0?$$

Distance and Spheres

Solution

This region represents a hemisphere of radius 4 with a hemisphere of radius 1 removed from it.