Parametric Equations and Polar Coordinates

10.3 Polar Coordinates

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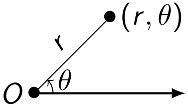
Calculus III

The Polar Coordinate System

Definition (Polar Coordinates)

The Polar Coordinate system consists of a marked point, O, called the **pole**, and a ray called the **polar axis**. A point, P, in the plane is specified in **polar coordinates** as a pair (r, θ), where r is the distance between O and P and θ is the angle (measured counterclockwise) between the polar axis and the line segment from O to P.



The Polar Coordinate System

Conventions

- A negative angle, $-\theta$, indicates an angle, θ, measured clockwise from the polar axis.

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- Negative angles can always be made positive by adding 2π.
- A point $(-r, \theta)$ is the same as the point $(r, \theta + \pi)$.

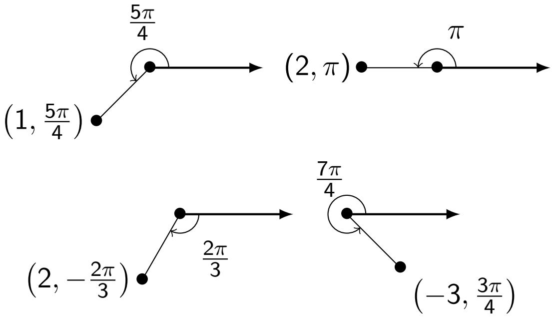
The Polar Coordinate System

Exercise

Plot the points

$$\left(1, \frac{5\pi}{4}\right) \qquad (2, 3\pi) \qquad \left(2, -\frac{2\pi}{3}\right) \qquad \left(-3, \frac{3\pi}{4}\right)$$

The Polar Coordinate System



Relationship between Polar and Cartesian Coordinates

Converting to Cartesian Coordinates

The point (r, θ) in Polar Coordinates has Cartesian Coordinates,

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

Relationship between Polar and Cartesian Coordinates

Converting to Polar Coordinates

The Polar Coordinates for the point (x, y) can be deduced from

$$r = \sqrt{x^2 + y^2} \qquad \tan(\theta) = \frac{y}{x}$$

provided $x \neq 0$.

Relationship between Polar and Cartesian Coordinates

Caution!

Using $\theta = \arctan(y/x)$ may result in **incorrect** polar coordinates!

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Example

The point $(-1, 1)$ lies on the circle of radius $r = \sqrt{2}$ and $\tan(\theta) = -1$. However, $\theta = \arctan(-1) = 7\pi/4$ gives polar coordinates $(\sqrt{2}, 7\pi/4)$ that **do not correspond to the Cartesian coordinates** $(-1, 1)$**!**

Relationship between Polar and Cartesian Coordinates

Remark

When $x = 0$,

$$\theta = \begin{cases} \dfrac{\pi}{2} & \text{if } y > 0 \\[2ex] \dfrac{3\pi}{2} & \text{if } y < 0 \end{cases}$$

Polar Curves

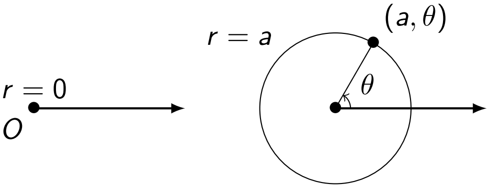
Definition

A **polar curve** is the set of all points in the plane with representation in polar coordinates, (r, θ), that satisfy an equation $r = f(\theta)$.

Polar Curves

Circles about the Origin

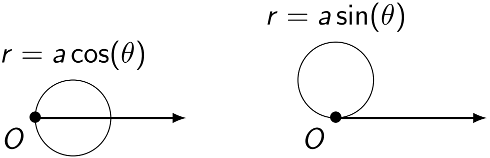
Assume $a \geq 0$. The polar curve $r = a$ is a circle of radius a when $a > 0$ and the degenerate circle O when $a = 0$.



Polar Curves

Circles Tangent to the Pole

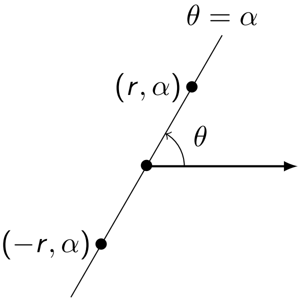
The polar curves $r = a\cos(\theta)$ and $r = a\sin(\theta)$ produce circles that are tangent to the pole.



Polar Curves

Lines through the Pole

For any angle α, the polar curve $\theta = \alpha$ is a line through the origin.

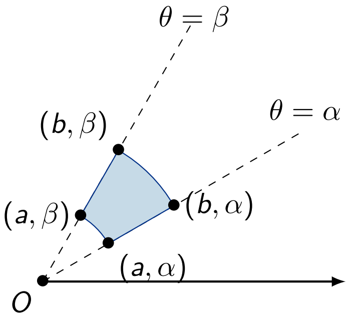


Polar Regions

Polar Rectangles

A **polar rectangle** is a region of the form

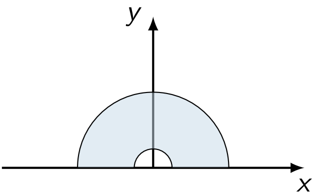
$$R = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$$



Polar Regions

Exercise

Describe the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ as a polar rectangle.



Polar Regions

Solution

This region is the polar rectangle

$$R = \{(r, \theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$$