Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

B. Farman

Mathematics and Statistics
Louisiana Tech University

Calculus III

Parametric Equations

Definition (Parametric Equations)

Suppose that x and y are both given as functions of a third variable, t, called a **parameter**, by the equations

$$x = f(t) \qquad\qquad y = g(t)$$

that are called **parametric equations**. The points $(x, y) = (f(t), g(t))$ in the plane trace out a **parametric curve**.

Parametric Equations

Example

The parametric equations

$$x(t) = t^2 - 2t \qquad\qquad y(t) = t + 1$$

cut out the shift of $x = y^2$ up 2 and left 1 by writing $t = y - 1$ and

$$\begin{aligned} x &= (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2 \\ &= y^2 - 4y + 3 = (y-2)^2 - 1 \end{aligned}$$

Parametric Equations

Exercise

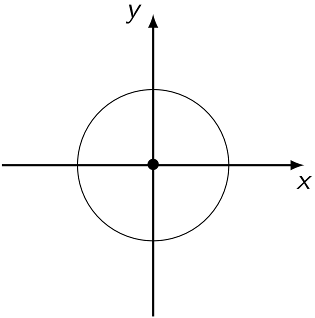
Sketch the parametric curve

$$x(t) = \cos(t) \qquad\qquad y(t) = \sin(t)$$

Parametric Equations

Solution

This is the unit circle



Parametric Equations

Example

Fix constants h, k, and r. Express the circle of radius r with center (h, k)

$$(x-h)^2 + (y-k)^2 = r^2$$

as a parametric curve.

Parametric Equations

Solution

Start from the unit circle $x(t) = \cos(t)$, $y(t) = \sin(t)$. First scale by r, then shift horizontally by h and vertically by k to obtain

$$x(t) = h + r\cos(t) \qquad\qquad y(t) = k + r\sin(t)$$

for $0 \leq t \leq 2\pi$.

Parametric Equations

Exercise

Fix constants a, b, c, and d. Assume $a \neq 0$. Sketch the parametric curve

$$x(t) = at + b \qquad\qquad y(t) = ct + d$$

Parametric Equations

Solution

Write

$$t = \frac{x-b}{a} \qquad y = c\left(\frac{x-b}{a}\right) + d = \frac{c}{a}(x-b) + d$$

to recognize this as the line through the point (b, d) with slope c/a.

Remark

In general,

$$x(t) = ax + b \qquad\qquad y(t) = cx + d$$

represents a line through (b, d). If $a = 0$ and $c \neq 0$, then it is the vertical line $x = b$. If $a = 0$ and $c = 0$, this curve is the degenerate line (b, d).

Parametric Equations

Vectors provide a convenient shorthand for parametric equations. Instead of expressing our curve as two parametric equations

$$x = f(t) \qquad\qquad y = g(t)$$

we could simply write $\langle f(t), g(t)\rangle$ or, frequently as a **vector-valued function**

$$\mathbf{v}(t) = \langle f(t), g(t)\rangle$$

Parametric Equations

Example

The line through $(2, 4)$ with slope 3 could be expressed as the equation

$$y - 4 = 3(x - 2)$$

or as the vector-valued function

$$\mathbf{v}(t) = \langle t + 2, 3t + 4 \rangle$$

Parametric Equations

It is natural to imagine parameterizing curves in space by introducing a third function

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and expressing this as a vector with parameterized coordinates:

$$\mathbf{v}(t) = \langle f(t), g(t), h(t) \rangle$$